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Marc ATLAN
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Risques, Options sur Hedge Funds et Produits Hybrides

Soutenue le 23 Janvier 2007, devant le jury composé de :

Mme Hélyette GEMAN (ESSEC - Birkbeck College), *co-directrice de thèse*
Mme Monique JEANBLANC (Université d'Evry), *rapporteur*
M. Damien LAMBERTON (Université de Marne-la-Vallée), *rapporteur*
M. Dilip MADAN (University of Maryland), *examinateur*
M. Gilles PAGÈS (Université Paris VI), *examinateur*
M. Marc YOR (Université Paris VI), *directeur de thèse*

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Introduction

0.1 Motivations

Cette thèse a deux objectifs principaux :

-Le premier est d'aborder des problèmes de modélisation liés à l'émergence de nouveaux types de produits dérivés. Ainsi, des questions telles que la dépendance entre le marché Actions et celui du Crédit ou celle entre le marché Actions et celui des Taux d'intérêt sont étudiées. La prise en compte de la nature spécifique des frais de gestion facturés par les hedge funds à leurs investisseurs est modélisée afin d'évaluer cet effet dans le pricing d'options sur les hedge funds eux-mêmes. Les différentes modélisations ainsi obtenues m'ont conduit à étudier la littérature sur les distributions des processus de Bessel et de diffusions qui en dérivent, la théorie des excursions du mouvement Brownien, les premiers temps de passage de ces processus, la théorie des changements de temps et les concepts de vraie martingale et/ou martingale locale stricte.

-Le second objectif est de s'interroger sur la notion de prime de risque sur les marchés d'options en faisant abstraction des modèles mathématiques sous-jacents. Pour cela, la théorie générale des processus et plus particulièrement les concepts d'immersion et de changement de mesure devraient permettre aux agents financiers de comprendre les risques non ou mal perçus par les modélisations usuelles.

0.2 Organisation et contribution

• Cette thèse est composée de cinq chapitres qui correspondent aux articles écrits entre Janvier 2004 et Avril 2006. Chaque chapitre peut, en principe, être lu indépendamment des autres. Toutefois, à bien des égards, les trois premiers chapitres se complètent. Tout d'abord, ils traitent tous les trois de problèmes liés aux modèles hybrides, ensuite ils utilisent des propriétés des processus de Bessel et enfin ils abordent des questions de volatilité locale, de volatilité stochastique ainsi que de volatilité locale stochastique. Sur la question des hybrides, le premier chapitre aborde l'impact de taux d'intérêt stochastiques sur l'évaluation d'options sur actions alors que les chapitres 2 et 3 (écrits avec Boris Leblanc) étudient la dépendance entre le marché des actions et celui du crédit. Enfin, ils visent à fournir des formules analytiques pour les différentes quantités mises en jeu ; nous reviendrons sur celles-ci dans les chapitres concernés ainsi que brièvement dans la section qui suit.

• Le chapitre 4 (écrit avec Hélyette Geman et Marc Yor), quant à lui, propose un modèle stochastique où l'actif étudié est un hedge fund et où l'on tient compte des frais de gestion et de performance. Il y est choisi un modèle de type Brownien géométrique ou encore de type Black Merton Scholes (1973) et l'on considère un drift qui est une fonction déterministe du temps et de la Valeur Actualisée Net (NAV) du fond. Dans ce contexte, nous sommes en mesure de fournir des formules quasi-analytiques pour des prix d'options vanille européennes. Le modèle ainsi construit est finalement une diffusion à volatilité constante mais à drift local où la forme particulière du drift permet d'obtenir une formule fermée pour la transformée de Laplace de la loi de la NAV. Il apparaît ainsi que mathématiquement les quatre premiers chapitres ont en commun l'étude de diffusions inhomogènes dont les dynamiques, telles qu'elles sont spécifiées, en font des modèles intéressants et nouveaux en mathématiques financières.

• Pour résumer la démarche suivie dans cette thèse, des produits nouveaux y sont étudiés et des risques généralement ignorés dans les modélisations classiques y sont caractérisés. Ceci peut être compris comme une illustration du principe d'adaptation de modèles connus à des problèmes nouveaux. Ayant choisi d'observer les modèles d'évaluation sous l'angle des risques qui sont pris en compte, l'on est naturellement amené à se demander ce que signifie valoriser un risque. Ceci nous ramène à l'idée de Markowitz (1952) qui formalise le fait qu'un risque est valorisé dès lors qu'il y a un excès de rendement. Dans le langage probabiliste des options, cela veut dire que la mesure de pricing sous la probabilité statistique et celle sous la probabilité risque-neutre diffèrent.

• Ainsi, le chapitre 5 écrit avec Hélyette Geman, Dilip Madan et Marc Yor examine à un niveau conceptuel illustré d'exemples provenant de propriétés fines du mouvement Brownien et des grossissements de filtration, la question des risques qui sont ou ne sont pas *pricés* dans une économie. On introduit diverses acceptions pour la notion de risque valorisé dans le chapitre 5.

0.3 Les contributions nouvelles

Dans les quatre paragraphes ci-dessous, nous donnons les idées maîtresses et les principaux résultats des cinq chapitres (le second paragraphe regroupe la discussion sur les chapitres 2 et 3) qui composent cette thèse.

0.3.1 Autour de la Volatilité Locale

Il s'agit ici de présenter succinctement les principales directions du chapitre 1 qui est en fait un article intitulé *Localizing volatilities*. Rappelons d'abord que le concept de volatilité locale remonte à Dupire (1994) et Derman et Kani (1994) et que si l'on considère un actif modélisé par la diffusion :

$$\frac{dS_t}{S_t} = r(t)dt + \sigma(t, S_t)dW_t$$

alors connaissant les prix des calls pour un continuum de maturités et de prix d'exercice, la fonction $(t, x) \mapsto \sigma(t, x)$ peut être définie implicitement par

$$\sigma^2(t, x) = \frac{\frac{\partial C}{\partial t} + xr(t)\frac{\partial C}{\partial x}}{\frac{1}{2}x^2\frac{\partial^2 C}{\partial x^2}}$$

A présent, considérons un modèle à volatilité stochastique générale de la forme

$$\frac{dS_t}{S_t} = r(t)dt + \sigma_t dW_t$$

où σ_t est un processus d'Itô continu. En se fondant sur les résultats de Gyöngy (1986), l'on obtient l'existence d'une équation différentielle stochastique Markovienne inhomogène telle que, à t fixé, la loi de la solution soit exactement la même que celle de S_t. De plus, on est capable de construire cette EDS et l'on obtient ainsi une détermination de la fonction $\sigma(t,x)$:

$$\sigma^2(t,x) = \mathbb{E}[\sigma_t^2 | S_t = x]$$

Maintenant qu'il est établi comment les modèles à volatilité stochastique et volatilité locale sont liés, l'on fournit une série d'exemples de modèles à volatilité stochastique construits à partir de processus de Bessel (type Heston (1993)) où l'on est en mesure de calculer explicitement la fonction volatilité locale. Le calcul de la volatilité locale est alors principalement basé sur la connaissance de la loi du couple (R_t^2, A_t) où $(R_t, t \geq 0)$ est un processus de Bessel et $A_t = \int_0^t R_s^2 ds$.

L'autre direction principale de ce chapitre est l'addition de taux d'intérêt stochastiques à la diffusion à laquelle le cours de l'actif obéit. Ainsi, la dynamique de l'action s'écrit :

$$\frac{dS_t}{S_t} = r_t dt + \sigma_t dW_t$$

où $(r_t, t \geq 0)$ est également un processus d'Itô continu. Nous étendons les résultats connus dans le cas de taux déterministes et obtenons l'existence d'une diffusion de la forme :

$$\frac{dX_t}{X_t} = r_t dt + \sigma(t, X_t) dW_t$$

telle que pour tout t fixé, les lois de S_t et X_t soient les mêmes. Pour cela, la fonction $\sigma(t,x)$ doit vérifier :

$$\sigma^2(t,x) = \frac{\mathbb{E}[\sigma_t^2 e^{-\int_0^t r_s ds} | S_t = x]}{\mathbb{E}[e^{-\int_0^t r_s ds} | S_t = x]}$$

D'autre part, l'équation implicite portant sur la volatilité locale peut être étendue au cas des taux stochastiques de la façon suivante :

$$\sigma^2(t,x) = \frac{\frac{\partial C}{\partial t} - x\mathbb{E}[e^{-\int_0^t r_s ds} r_t \mathbf{1}_{\{S_t > x\}}]}{\frac{x^2}{2}\frac{\partial^2 C}{\partial x^2}}$$

D'autres constructions de volatilité locale à partir de modèles à volatilité stochastique sont proposées dans ce chapitre et une réécriture et discussion de l'équation implicite ci-dessus sont fournies.

0.3.2 Valorisation de Produits Dérivés Hybrides Action et Crédit

Cette section résume le travail élaboré en collaboration avec Boris Leblanc qui a donné lieu à deux chapitres (2 et 3) de cette thèse et ainsi qu'à deux articles dont le premier intitulé *Time-Changed Bessel Processes and Credit Risk* est soumis à Mathematical Finance et dont le second intitulé *Hybrid Equity-Credit Modelling* a été publié dans Risk Magazine. La problématique posée est la modélisation du cours de l'action d'une compagnie en tenant compte de son risque de faillite, lui-même un actif financier coté sur le marché obligataire et sur celui du crédit. Un standard de marché depuis quelques années pour modéliser le cours d'une action sous cette contrainte est le modèle à probabilité locale de défaut (voir par exemple Davis et Lischka (2002)) où le défaut que l'on définit comme le fait que l'action vaille zéro, est calibré sur un processus à sauts destiné à engendrer la faillite et dont la partie continue est quant à elle présente pour calibrer la surface de volatilité implicite dans des zones où les options sont cotées.

Le problème de ce type de modélisation est qu'il ne tient pas compte des scénarios de marché où le cours de l'actif baisse lentement en se dirigeant vers la faillite sans pour autant faire de saut significatif et en entraînant une hausse des spreads de crédit. Ayant ce type d'inconvénient à l'esprit, nous nous intéressons au modèle Constant Elasticity of Variance de Cox (1975) pour son aptitude à générer des trajectoires qui atteignent 0 en un temps fini. Cette propriété résulte du fait que le CEV peut être vu comme une puissance de Bessel changé de temps. On notera que dans la même année Albanese et Chen (2005), Campi, Polbennikov et Sbuelz (2005) et Carr et Linetsky (2005) ont étudié ce modèle dans le même objectif. Rappelons maintenant la dynamique de ce processus :

$$\frac{dS_t}{S_t} = rdt + \sigma S_t^{\alpha-1} dW_t$$

Ainsi le chapitre 2 propose une étude détaillée du processus CEV ; on y appliquera les résultats de cette étude à la valorisation d'options vanilles, de credit default swaps (CDS) et d'equity default swaps (EDS) pour lesquels on obtient des formules analytiques. L'on étend par la suite ce modèle en y ajoutant une volatilité stochastique dont le mouvement brownien qui l'engendre est supposé indépendant de celui qui engendre le cours de l'action :

$$\frac{dS_t}{S_t} = rdt + \sigma_t S_t^{\alpha-1} dW_t$$

Dans cette configuration, l'on est en mesure d'obtenir des formules analytiques pour les prix d'options et pour les CDS conditionnellement à la connaissance de la loi pour tout t de la quantité

$$H_t = (1-\alpha)^2 \int_0^t \sigma_s^2 e^{-2(1-\alpha)rs} ds$$

L'addition de volatilité stochastique permet de générer plus ou moins de skew et de smile à la surface de volatilité implicite. Maintenant afin de pouvoir décorréler la probabilité de défaut de la "skewness", on corrèle les mouvements de l'action à ceux de sa volatilité. De nombreux exemples où des calculs analytiques sont possibles sont présentés.
Le chapitre 3 illustre de façon appliquée ces propriétés mathématiques des processus CEV et implicitement des processus de Bessel, et également les arguments financiers empiriques discutés plus haut et qui y sont détaillés.

0.3.3 Frais de gestion et de performance et Options sur Hedge Funds

Lors de l'évaluation d'options sur Hedge Funds, sous réserve que l'on puisse se couvrir, il est important de considérer les différents frais (importants) liés à la détention de participations dans un hedge fund. En effet, l'investisseur est soumis à des frais de gestion qui sont fixes et de l'ordre de 1% à 2% du montant investi mais il est également soumis à des frais de performance suivant la High-Water Mark rule, qui sont de l'ordre de 15% à 20% de la performance annuelle du hedge fund. Cette question est traitée dans le chapitre 4 qui, à l'origine, est un article écrit avec Hélyette Geman et Marc Yor intitulé *Options on Hedge Funds under the High-Water Mark Rule* soumis à Quantitative Finance.

Afin de tenir compte de ces différents frais, nous proposons le modèle suivant pour régir la valeur nette actualisée S :

$$\frac{dS_t}{S_t} = (r + \alpha - c - f(S_t))dt + \sigma dW_t$$

où r est le taux sans risque, c représente les frais de gestion, α l'excès de rendement du fond par rapport au marché, f est la fonction modélisant le high-water mark :

$$f(s) = \mu a \ \mathbf{1}_{\{s>H\}}$$

et μ est le rendement statistiquement observé, a le pourcentage prélevé sur la performance et H le niveau par rapport auquel cette performance est mesurée.

Relativement à ce modèle, nous sommes en mesure de calculer les prix de calls européens. Ainsi, l'on démontre que le calcul des prix d'options vanille repose essentiellement sur la connaissance de la quantité

$$g(t) = \mathbb{E}\big[h(W_t)\exp(\lambda L_t)\exp(-\mu A_t^+ - \nu A_t^-)\big]$$

où W_t est un mouvement brownien, L_t est son temps local en 0 et A_t^+ etA_t^- respectivement les temps passés positivement et négativement jusqu'au temps t. Il est remarquable que la variable tri-dimensionnelle mise en jeu (W_t, L_t, A_t^+) joue un rôle crucial dans l'étude de la loi de l'Arcsinus étudié par exemple dans Karatzas et Shreve (1991). Il est relativement clair que le calcul de la transformée de Laplace de g peut être fait, par exemple, via un détour par la théorie des excursions du mouvement brownien (voir Jeanblanc, Pitman et Yor (1997) ou Revuz et Yor (2005)) et l'on démontre ainsi que pour tout θ suffisamment grand

$$\int_0^\infty dt e^{-\frac{\theta}{2}t} g(t) = 2\frac{\left(\int_0^\infty dx e^{-x\sqrt{\theta+2\mu}} h(x) + \int_0^\infty dx e^{-x\sqrt{\theta+2\nu}} h(-x)\right)}{\sqrt{\theta+2\mu} + \sqrt{\theta+2\nu} - 2\lambda}$$

0.3.4 Prime de Risque et Théorie Générale des Processus

Une question très importante en finance de marché est l'identification des risques dans l'évaluation d'actifs financiers. Depuis Fama (1970), l'on dit qu'un risque est valorisé dans l'évaluation d'un actif financier dès lors que la covariance entre le risque considéré et l'actif observé est différente de zéro. Une conséquence importante de cette vision est qu'un risque n'est pas valorisé dans une économie s'il n'est pas corrélé aux actifs qui la composent. Un peu plus récemment dans la littérature sur les produits dérivés (Harrison et Kreps (1979), Harrison et Pliska (1981)), un risque est *pricé* si les valorisations de l'actif d'Arrow Debreu associé à ce risque ont des espérances différentes sous la probabilité statistique et sous la probabilité risque neutre. Les deux approches sont cohérentes dans la mesure où un risque est *pricé* dès qu'il impacte les excès de rendements. Mathématiquement, la corrélation entre les processus de l'économie et le risque donné précisément par le changement de mesure n'est pas un critère suffisant pour assurer qu'un risque est *pricé* ou non.

Selon ces considérations, le chapitre 5 rédigé avec Hélyette Geman, Dilip Madan et Marc Yor a donné lieu à un article intitulé *Correlation and the Pricing of Risks* accepté dans Annals of Finance. Nous y fournissons entre autres des exemples de risques *pricés* et de corrélation nulle. Après l'introduction de nouveaux concepts et une étude détaillée de la notion de risque, laquelle est considéré aussi bien au niveau des variables aléatoires, des processus continus que des filtrations nous sommes capables de montrer que l'absence d'excès de rendement et une corrélation nulle pour des processus continus sont équivalents à condition de se placer dans une filtration étendue que l'on appellera *self sufficient*. Cette filtration construite en détail dans le chapitre 5 prend en compte les trajectoires de processus permettant de prédire l'évolution du risque considéré. Nous démontrons que la *self sufficiency* d'une filtration dépend de la mesure de probabilité sous laquelle elle est étudiée et nous examinons les changements de mesure de probabilité pour lesquelles la *self sufficiency* est préservée.

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Chapitre 1

Localizing Volatilities

[*Submitted*[*]]

We propose two main applications of Gyöngy (1986)'s construction of inhomogeneous Markovian stochastic differential equations that mimick the one-dimensional marginals of continuous Itô processes. Firstly, we prove Dupire (1994) and Derman and Kani (1994)'s result. We then present Bessel-based stochastic volatility models in which this relation is used to compute analytical formulas for the local volatility. Secondly, we use these mimicking techniques to extend the well-known local volatility results to a stochastic interest rates framework.

1.1 Introduction

It has been widely accepted for at least a decade that the option pricing theory of Black and Scholes (1973) and Merton (1973) has been inconsistent with option prices. Actually, the model implies that the informational content of the option surface is one dimensional which means that one could construct the prices of options at all strikes and maturities from the price of any single option. It has also been shown that unconditional returns show excess kurtosis and skewness which are inconsistent with normality. Special attention was given to implied volatility smile or skew, but research has concentrated on implied Black and Scholes volatility since it has become the unique way to price vanilla options. Accordingly, option prices are often quoted by their implied volatility. Nevertheless, this method is unsuitable for more complicated exotic options and options with early exercise features. To explain in a self-consistent way why options with different strikes and maturities have different implied volatilities or what one calls the volatility smile, one could use stochastic volatility models (eg. Heston (1993) or Hull and White (1987))

Given the computational complexity of stochastic volatility models and the difficulty of fitting their parameters to the market prices of vanilla options, practitioners found a simpler way to price exotic options consistently with the volatility smile by using local volatility models as introduced by Dupire (1994) and Derman and Kani (1994). Local volatility models have the advantage to fit the implied volatility surface; hence, when pricing an exotic option, one feels comfortable hedging

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through the stock and vanilla options markets.

In the last twenty years, academics and practitioners have been primarily interested in building models that describe well the behavior of an asset whether it is equity, FX, Credit, Fixed-Income or Commodities and very rarely models that specify any cross-asset dependency. For all cross-asset derivative products, this dependency modifies the model one should use or at least the calibration procedure. Certainly, models that incorporate a dependency on other asset classes than a specific underlying need to be recalibrated as soon as the other asset classes become random, in particular in the fast growing hybrid industry where it is necessary to model several assets.

The remainder of the paper is organized as follows. Section 2 recalls preliminary results on Bessel processes and states mimicking properties of continuous Itô processes exhibited by Gyöngy (1986) and Krylov (1985). Section 3 recalls well-known results of Dupire (1994) and Derman and Kani (1994) on local volatility, gives a proof of the existence of a local volatility model that mimicks a stochastic volatility one based on Gyöngy (1986) theorem. Section 4 gives examples of stochastic volatility models where a local volatility can be computed. Those examples are based on remarkable properties of Bessel processes such as scaling properties. In order to extend the class of volatility models (where closed-form formulas can be obtained), we propose a general framework in which the volatility diffusion is a general deterministic time and space transformation of Bessel processes. Analytical computations are proposed in cases where the volatility diffusion is independent from the stock price diffusion as well as in cases where they are correlated. Section 5 applies the results of Section 3 to the case of stochastic interest rates and more generally shows how Gyöngy (1986) theorem can be applied to construct a local volatility model in a deterministic interest rate framework, starting from a stochastic volatility model with stochastic rates. Finally, Section 6 concludes our work and presents an important open question on mimicking the laws of Itô processes.

1.2 Preliminary Mathematical Results

1.2.1 Bessel and CIR Processes

Let $(R_t, t \geq 0)$ denote a Bessel process with dimension δ, starting from 0 and $(\beta_t, t \geq 0)$ an independent brownian motion from $(B_t, t \geq 0)$ the driving brownian motion. Let us recall that R_t^2 solves the following SDE:

$$dR_t^2 = 2R_t dB_t + \delta dt$$

and let us now define :

$$I_t = \int_0^t R_s d\beta_s \qquad and \qquad A_t = \int_0^t R_s^2 ds$$

Then, the one-dimensional marginals of (A_t, I_t) are at least in theory well-identified, via Fourier-Laplace expressions, and are closely related with the so-called Lévy area formula (see Lévy (1950), Williams (1976), Gaveau (1977), Yor (1980), Chapter 2 of Yor (1992) and many other references). Here we simply recall, for our purposes the formulae:
$\forall(\alpha, \beta) \in \mathbb{R}^2$

$$\mathbb{E}\Big[\exp\big(i\alpha I_t - \frac{\beta^2}{2}A_t\big)\Big] = \big(\cosh(t\sqrt{\alpha^2+\beta^2})\big)^{-\frac{\delta}{2}} \tag{1.1}$$

as well as:
$\forall (a,b) \in \mathbb{R}_+ \times \mathbb{R}$

$$\mathbb{E}\Big[\exp\big(-aR_t^2 - \frac{b^2}{2}A_t\big)\Big] = \big(\cosh(bt) + \frac{2a}{b}\sinh(bt)\big)^{-\frac{\delta}{2}} \tag{1.2}$$

a formula that we shall use later. Some developments for the law of A_t are given, e.g. in Pitman and Yor (2003).
For a Bessel process of dimension δ starting at x, one gets the following formula:
$\forall (a,b) \in \mathbb{R}_+ \times \mathbb{R}$

$$\mathbb{E}_x\Big[\exp\big(-aR_t^2 - \frac{b^2}{2}A_t\big)\Big] = \big(\cosh(bt) + \frac{2a}{b}\sinh(bt)\big)^{-\frac{\delta}{2}} \times \exp\Big(-\frac{x^2 b}{2}\frac{\sinh(bt) + \frac{2a}{b}\cosh(bt)}{\cosh(bt) + \frac{2a}{b}\sinh(bt)}\Big)$$

Let us now present a scaling property of the Bessel process with respect to conditioning, which is important in the sequel.

Proposition 1.2.1 *For any Bessel process R_t with dimension δ, we have:*

$$\mathbb{E}\Big[R_t^2 | \int_0^t R_s^2 ds\Big] = \frac{2}{t}\int_0^t R_s^2 ds \tag{1.3}$$

Remark 1.2.2 *This result is in fact a very particular case of a more general result involving only the scaling property of the process $(R_t^2, t \geq 0)$, see, e.g, Pitman and Yor (2003). But, for the sake of completeness, we shall give a direct proof of (1.3) below:*

Proof. From the scaling property of $(R_t^2, t \geq 0)$, we deduce that for every $f \in \mathcal{C}^1(\mathbb{R}, \mathbb{R}_+)$, with bounded derivative, we have:

$$\mathbb{E}\Big[f\Big(\int_0^t R_s^2 ds\Big)\Big] = \mathbb{E}\Big[f\Big(t^2 \int_0^1 R_s^2 ds\Big)\Big]$$

We then differentiate both sides with respect to t, to obtain:

$$\begin{aligned}\mathbb{E}\Big[f'\Big(\int_0^t R_s^2 ds\Big)R_t^2\Big] &= \mathbb{E}\Big[f'\Big(t^2\int_0^1 R_s^2 ds\Big)(2t)\int_0^1 R_s^2 ds\Big] \\ &= \mathbb{E}\Big[f'\Big(\int_0^t R_s^2 ds\Big)\frac{2}{t}\int_0^t R_s^2 ds\Big]\end{aligned}$$

Since this identity is true for every bounded Borel function f', the identity (1.3) follows. ■

Remark 1.2.3 *We now check that formula (1.3) can be obtained directly as a consequence of formula (1.2): differentiating (1.2) both sides with respect to a and taking $a = 0$, we obtain:*

$$\mathbb{E}\Big[R_t^2 \exp\big(-\frac{b^2}{2}A_t\big)\Big] = \frac{\delta}{(\cosh(bt))^{\frac{\delta}{2}+1}}\big(\frac{1}{b}\sinh(bt)\big)$$

while, taking $a = 0$ in (1.2), and differentiating both sides with respect to b, we get:

$$b\mathbb{E}\Big[A_t \exp\big(-\frac{b^2}{2}A_t\big)\Big] = \frac{\delta t}{2(\cosh(bt))^{\frac{\delta}{2}+1}}\sinh(bt)$$

and the identity (1.2) follows from the comparison of these last two equations.

A reason why squared Bessel processes play an important role in financial mathematics is that they are connected to models used in finance. One of these models is the Cox, Ingersoll and Ross (1985) CIR family of diffusions which are solutions of the following kind of SDEs:

$$dX_t = (a - bX_t)dt + \sigma\sqrt{|X_t|}dW_t \tag{1.4}$$

with $X_0 = x_0 > 0$, $a \in \mathbb{R}_+$, $b \in \mathbb{R}$, $\sigma > 0$ and W_t a standard brownian motion. This equation admits a unique strong (that is to say adapted to the natural filtration of W_t) solution that takes values in $\mathbb{R}_+$.

One is now interested in the representation of a CIR process in terms of a time-space transformation of a Bessel process:

Lemma 1.2.4 *A CIR Process X_t solution of equation (2.1) can be represented in the following form:*

$$X_t = e^{-bt}R^2_{\frac{\sigma^2}{4b}(e^{bt}-1)} \tag{1.5}$$

where R denotes a Bessel process starting from x_0 at time $t = 0$ of dimension $\delta = \frac{4a}{\sigma^2}$

Proof. This lemma results from simple properties of squared Bessel processes that can be found in Revuz and Yor (2001), Pitman and Yor (1980, 1982). ■

This relation has been widely used in finance, for instance in Geman and Yor (1993) or Delbaen and Shirakawa (1996).

1.2.2 Mimicking Theorems

A common topic of interest of Krylov and Gyöngy respectively in Krylov (1985) and Gyöngy (1986) is the construction of stochastic differential equations whose solutions mimick certain features of the solutions of Itô processes. The construction of Markov martingales that have specified marginals was studied by Madan and Yor (2002). Bibby, Skovgaard and Sørensen (2005) as well as Bibby and Sørensen (1995) proposed construction of diffusion-type models with given marginals.

Let us now consider an Itô differential equation of the form:

$$\xi_t = \int_0^t \delta_s dW_s + \int_0^t \beta_s ds \tag{1.6}$$

where W_t is a $\mathcal{F}_t$-Brownian motion of dimension k, $(\delta_t)_{t\in\mathbb{R}_+}$ and $(\beta_t)_{t\in\mathbb{R}_+}$ are bounded $\mathcal{F}_t$-adapted processes that belong respectively to $\mathbf{M}_{n,k}(\mathbb{R})$, the space of $n \times k$ real matrices and to $\mathbb{R}^n$.

Definition 1.2.5 (Green Measure) *Considering two stochastic processes X_t, valued in $\mathbb{R}^n$ and γ_t, with $\gamma_t > 0$, one defines the Green measure $\mu_{X,\gamma}$ by:*

$$\mu_{X,\gamma}(\Gamma) = \mathbb{E}\Big[\int_0^\infty \mathbf{1}_\Gamma(X_t) \exp\big(-\int_0^t \gamma_s ds\big) dt\Big] \tag{1.7}$$

where Γ is any borel set of $\mathbb{R}^n$

Remark 1.2.6 *The stochastic process γ_t is called the killing rate*

Theorem 1.2.7 (Krylov) *If ξ_t is an Itô process defined as previously and satisfying the uniform ellipticity condition: $\exists \lambda \in \mathbb{R}_+^*$ such as $\delta\delta^* \geq \lambda I_n$*
as well as the lower boundedness condition: $\exists \alpha \in \mathbb{R}_+$ such as $\gamma > \alpha$,
then there exist deterministic functions $\sigma : \mathbb{R}^n \to \mathbf{M}_n(\mathbb{R})$, $b : \mathbb{R}^n \to \mathbb{R}^n$ and $c : \mathbb{R}^n \to \mathbb{R}_+$ such that the following SDE:

$$\begin{aligned} dx_t &= \sigma(x_t)dW_t + b(x_t)dt \\ x_0 &= 0 \end{aligned}$$

has a weak solution satisfying:

$$\begin{aligned} &\forall \Gamma \in \mathcal{B}(\mathbb{R}^n) \\ \mathbb{E}\Big[\int_0^\infty \mathbf{1}_\Gamma(\xi_t) \exp\big(-\int_0^t \gamma_s ds\big) dt\Big] &= \mathbb{E}\Big[\int_0^\infty \mathbf{1}_\Gamma(x_t) \exp\big(-\int_0^t c(x_s) ds\big) dt\Big] \\ ie : \mu_{\xi,\gamma}(\Gamma) &= \mu_{X,c}(\Gamma) \end{aligned}$$

Proof. See Krylov (1985) ■

Definition 1.2.8 (Weak Solution) *The stochastic differential equation*

$$dX_t = f(t, X_t)dW_t + g(t, X_t)dt \tag{1.8}$$
$$X_0 = 0 \tag{1.9}$$

is said to have a weak solution if there exist a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and an $\mathcal{F}_t$-Brownian motion with respect to which there exists a $\mathcal{F}_t$-adapted stochastic process $\overline{X}_t$ that satisfies (1.8) and (1.9).

A natural question asked and answered by Gyöngy is whether it is possible to find the solution of an SDE with the same one-dimensional marginal distributions as an Itô process. The answer is stated below:

Theorem 1.2.9 (Gyöngy) *If ξ_t is an Itô process satisfying the uniform ellipticity condition: $\exists \lambda \in \mathbb{R}_+^*$ such as $\delta\delta^* \geq \lambda I_n$*
then there exist bounded measurable functions $\sigma : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbf{M}_{n,n}(\mathbb{R})$ and $b : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ defined by:

$$\begin{aligned} &\forall (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n \\ \sigma(t, x) &= \Big(\mathbb{E}\big[\delta_t \delta_t^* | \xi_t = x\big]\Big)^{\frac{1}{2}} \\ b(t, x) &= \mathbb{E}\big[\beta_t | \xi_t = x\big] \end{aligned}$$

such that the following SDE:

$$\begin{aligned} dx_t &= \sigma(t, x_t)dW_t + b(t, x_t)dt \\ x_0 &= 0 \end{aligned}$$

has a weak solution with the same one-dimensional marginals as ξ.

Proof. See Gyöngy (1986) ■

Remark 1.2.10 *Two kinds of mimicking features of a general Itô process were illustrated in this section . With Krylov, we were able to construct a Markov homogeneous process solution of an SDE, that has the same Green measure than the Itô process. Using Gyöngy's results, we were able to build a time-inhomogeneous Markov process solution of an SDE that has the same one-dimensional marginals as the Itô process.*

A possible extension to the above mimicking property is to consider a real Itô process ξ driven by a multidimensional Brownian motion and obtain a new mimicking result useful for the remainder of the paper; the proof is straightforward from Gyöngy (1986) proof. Let ξ be as follows :

$$\xi_t = \int_0^t < \delta_s, dW_s > + \int_0^t \beta_s ds \tag{1.10}$$

where W_t is a $\mathcal{F}_t$-Brownian motion of dimension k, $(\delta_t)_{t\in\mathbb{R}_+}$ and $(\beta_t)_{t\in\mathbb{R}_+}$ are bounded $\mathcal{F}_t$-adapted processes that belong respectively to $\mathbb{R}^k$ and to $\mathbb{R}$.

Theorem 1.2.11 *If ξ_t is an Itô process defined as in (1.10) satisfying the uniform ellipticity condition: $\exists\lambda \in \mathbb{R}_+^*$ such as $\delta\delta^* \geq \lambda$*
then there exist bounded measurable functions $\sigma : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ and $b : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ defined by:

$$\forall (t, x) \in \mathbb{R}_+ \times \mathbb{R}$$

$$\begin{aligned} \sigma(t, x) &= \left(\mathbb{E}\big[\delta_t\delta_t^*|\xi_t = x\big]\right)^{\frac{1}{2}} \\ b(t, x) &= \mathbb{E}\big[\beta_t|\xi_t = x\big] \end{aligned}$$

such that the following SDE:

$$\begin{aligned} dx_t &= \sigma(t, x_t)dW_t + b(t, x_t)dt \\ x_0 &= 0 \end{aligned}$$

has a weak solution with the same one-dimensional marginals as ξ.

1.3 Generalities on Local Volatility

1.3.1 Fokker-Planck Equation

Let us assume that:

$$\frac{dS_t}{S_t} = r(t)dt + \sigma(t, S_t)dW_t \tag{1.11}$$

where r and σ are deterministic functions, σ is usually called the local volatility. Under the local volatility dynamics, option prices satisfy the following PDE:

$$\frac{\partial V}{\partial t} + \frac{\sigma^2(t, S_t)}{2} S^2 \frac{\partial^2 V}{\partial S^2} + r(t)S\frac{\partial V}{\partial S} - r(t)V = 0 \tag{1.12}$$

and terminal condition $V(S,T) = C(S,T) = PayOff_T(S)$.
If we consider call options, we would get $V(S,T) = (S-K)_+$. It has been proved that one can obtain a forward PDE for $C(K,T)$ instead of fixing (K,T) and obtaining a backward PDE for $C(S,t)$. To get the Forward PDE equation, one could just differentiate (1.12) twice with respect to the strike K and then get the same PDE, with variable $\phi = \frac{\partial^2 C}{\partial K^2}$ and terminal condition $\delta(S-K)$. ϕ is the transition density of S and is also the Green function of (1.12). It follows that ϕ as a function of (K,T) satisfies the Fokker-Planck PDE:

$$\frac{\partial \phi}{\partial T} - \frac{\partial^2}{\partial K^2}\big(\frac{\sigma^2(T,K)}{2}K^2\phi\big) + r(T)\frac{\partial}{\partial K}(K\phi) + r(T)K = 0$$

Now, integrate twice this equation taking into account the boundary conditions, one obtains the Forward Parabolic PDE equation:

$$\frac{\partial C}{\partial T} - \frac{\sigma^2(T,K)}{2}K^2\frac{\partial^2 C}{\partial K^2} + r(T)K\frac{\partial C}{\partial K} = 0 \tag{1.13}$$

with initial condition $C(K,0) = (S_0 - K)_+$. Hence, one obtains Dupire (1994) equation

$$\sigma^2(T,K) = \frac{\frac{\partial C}{\partial T} + r(T)K\frac{\partial C}{\partial K}}{\frac{1}{2}K^2\frac{\partial^2 C}{\partial K^2}} \tag{1.14}$$

Moreover, if one expresses the option price as a function of the forward price, one would write a simpler expression:

$$\sigma^2(T,K,S_0) = \frac{\frac{\partial C}{\partial T}}{\frac{1}{2}K^2\frac{\partial^2 C}{\partial K^2}}$$

where C is now a function of (F_T, K, T) with $F_T = S_0 \exp\big(\int_0^T r(s)ds\big)$.

1.3.2 Matching Local and Stochastic Volatilities

A stock price diffusion with a stochastic volatility is one of the following form:

$$\frac{dS_t}{S_t} = r(t)dt + \sqrt{V_t}dW_t \tag{1.15}$$

where V_t is a stochastic process, solution of an SDE and $r(t)$ is a deterministic function of time. (We do not yet discuss the dependence of the stock price and volatility processes, also called Leverage effect)
One can find a relation between the local volatility and a stochastic volatility. First, one applies Tanaka's formula to the stock price process:

$$\begin{aligned} e^{-\int_0^t r(s)ds}(S_t - x)_+ &= (S_0 - x)_+ - \int_0^t r(u)e^{-\int_0^u r(s)ds}(S_u - x)_+ du \\ &+ \int_0^t e^{-\int_0^u r(s)ds}\mathbf{1}_{\{S_u > x\}} dS_u + \frac{1}{2}\int_0^t e^{-\int_0^u r(s)ds} dL_u^x(S) \end{aligned}$$

Assuming that $(e^{-\int_0^t r(s)ds}S_t, t \geq 0)$ is a true martingale, then
$(\int_0^t \mathbf{1}_{\{S_u > x\}} d(e^{-\int_0^u r(s)ds}S_u), t \geq 0)$ is a martingale and one gets:

$$\begin{aligned} \mathbb{E}[e^{-\int_0^t r(s)ds}(S_t - x)_+] &= \mathbb{E}[(S_0 - x)_+] + x\int_0^t \mathbb{E}[r(u)e^{-\int_0^u r(s)ds}\mathbf{1}_{\{S_u > x\}}]du \\ &+ \frac{1}{2}\mathbb{E}\left[\int_0^t e^{-\int_0^u r(s)ds} dL_u^x(S)\right] \end{aligned}$$

Then, differentiating the previous relation and using Fubini theorem, one obtains:

$$d_t C(t,x) = x\mathbb{E}[r(t)e^{-\int_0^t r(s)ds}\mathbf{1}_{\{S_t > x\}}]dt + \frac{1}{2}\mathbb{E}[e^{-\int_0^t r(s)ds} dL_t^a(S)] \tag{1.16}$$

where $C(t,x) = \mathbb{E}[e^{-\int_0^t r(s)ds}(S_t - x)_+]$ Using a classical characterization of the local time of any continuous semi-martingale:

$$L_t^x(S) = \lim_{\epsilon \to 0}\frac{1}{\epsilon}\int_0^t \mathbf{1}_{\{x \leq S_s < x+\epsilon\}} d < S, S >_s \tag{1.17}$$

one gets with a permutation of the differentiation and the expectation:

$$d_t C(t,x) = x\mathbb{E}[r(t)e^{-\int_0^t r(s)ds}\mathbf{1}_{\{S_t > x\}}]dt + \frac{1}{2}\lim_{\epsilon \to 0}\mathbb{E}[\frac{1}{\epsilon}\mathbf{1}_{\{x \leq S_t < x+\epsilon\}} e^{-\int_0^t r(s)ds} V_t S_t^2]dt \tag{1.18}$$

as a result of $d < S, S >_t = V_t S_t^2 dt$. Now, one may write using conditional expectations and the fact that interest rates are assumed to be deterministic, the following identity:

$$\mathbb{E}[V_t S_t^2 \mathbf{1}_{\{x \leq S_t < x+\epsilon\}}] = \mathbb{E}\left[\mathbb{E}[V_t | S_t] S_t^2 \mathbf{1}_{\{x \leq S_t < x+\epsilon\}}\right]$$

From there assuming regularity conditions on $q_t(x)$ and $\mathbb{E}[V_t|S_t]$, one easily obtains:

$$\begin{aligned} \lim_{\epsilon \to 0}\frac{1}{\epsilon}\mathbb{E}[V_t S_t^2 \mathbf{1}_{\{x \leq S_t < x+\epsilon\}}] &= \lim_{\epsilon \to 0}\frac{1}{\epsilon}\mathbb{E}\left[\mathbb{E}[V_t|S_t] S_t^2 \mathbf{1}_{\{x \leq S_t < x+\epsilon\}}\right] \\ &= \mathbb{E}[V_t | S_t = x] x^2 q_t(x) \end{aligned}$$

where $q_t(x)$ is the value of the density of S_t in x. Since Breeden and Litzenberger (1978), it is well known that $\frac{\partial^2 C(t,x)}{\partial x^2} = e^{-\int_0^t r(s)ds} q_t(x)$. It is also known that $\frac{\partial C}{\partial x} = -\mathbb{E}[e^{-\int_0^t r(s)ds}\mathbf{1}_{\{S_t > x\}}]$ One finally may write:

$$\frac{\partial C}{\partial t} + xr(t)\frac{\partial C}{\partial x} = \mathbb{E}[V_t | S_t = x]\frac{1}{2}x^2\frac{\partial^2 C}{\partial x^2} \tag{1.19}$$

Comparing equation (1.14) and the above equation, one may obtain an equation that relates local and stochastic volatility models

$$\sigma^2(t,x) = \mathbb{E}[V_t|S_t = x] \tag{1.20}$$

Hence, we have proven that if there exists a local volatility such that the one-dimensional marginals of the stock price with the implied diffusion are the same as the ones of the stock price with the stochastic volatility, then the local volatility satisfies equation (1.20).

Another way to prove this relation is to apply Gyöngy (1986) result. Since the stock price dynamics with a stochastic volatility given by equation (1.15) and the ones with the local volatility given by equation (1.11) must have the same one-dimensional marginals, one can apply Gyöngy Theorem: assuming that there exists $\lambda \in \mathbb{R}_+^*$ such that $S^2V \geq \lambda$ we get the well-known relation between the local and the stochastic volatilities:

$$\sigma(t, S_t = x) = \Big(\mathbb{E}\big[V_t|S_t = x\big]\Big)^{\frac{1}{2}}$$

It is important to notice that Gyöngy gives us the existence of such a diffusion in addition to provide an explicit way to construct it. More generally, assuming just that the volatility process is a general continuous semi-martingale, one can also get the same result, and a justification for the use of local volatility models. Hence, we obtain an illustration of Gyöngy's result in a finance framework. Moreover, it is shown that one can get the relation (1.20) without using the Forward PDE equation.

As a first remark, we should notice that if we choose V_t such as $\sqrt{V_t} = \sigma(t, S_t)$, we then obtain another direct proof of equation (1.14).

As a second remark, we can prove that if $(\widetilde{S}_t = e^{-\int_0^t r(s)ds} S_t), t \geq 0)$ is a strict local martingale (which is studied in Cox and Hobson (2005) who named this market situation a bubble), then

$$\begin{aligned}
\mathbb{E}[\int_0^t \mathbf{1}_{\{S_u > x\}} d\widetilde{S}_u] &= \mathbb{E}[\widetilde{S}_t - S_0] - \mathbb{E}[\int_0^t \mathbf{1}_{\{S_u \leq x\}} d\widetilde{S}_u] \\
&= \mathbb{E}[\widetilde{S}_t - S_0]
\end{aligned}$$

since using Madan and Yor (2006), $(\int_0^t \mathbf{1}_{\{S_u \leq x\}} d(e^{-\int_0^u r(s)ds} S_u), t \geq 0)$ is a square integrable martingale. Hence, defining

$$c_{\widetilde{S}}(t) = \mathbb{E}[S_0 - \widetilde{S}_t]$$

and assuming that $c_{\widetilde{S}}$ is a continuously differentiable function, one obtains an extension of equation (1.14) that is a generalization to the case of strict local martingales. This equation writes

$$\sigma^2(t,x) = \frac{\frac{\partial C}{\partial t} + xr(t)\frac{\partial C}{\partial x} + c'_{\widetilde{S}}(t)}{\frac{1}{2}x^2\frac{\partial^2 C}{\partial x^2}}$$

1.4 Applications to the Heston (1993) model and Extensions

1.4.1 The Simplest Heston Model

The aim of this paragraph is now to compute the local volatility not by excerpting it from the option prices (see for instance Derman and Kani (1994)) but by applying Gyöngy's theorem.

Among the possible choices of stochastic volatility models, we will consider the simplest one, given by the following SDE:

$$\frac{dS_t}{S_t} = W_t dB_t \tag{1.21}$$
$$S_0 = 1$$

where (W_t) and (B_t) are two independent one-dimensional Brownian motions starting at 0. We do not consider any drift term in our stock diffusion as we look at the forward price dynamics that are driftless by construction.

To make our discussion a little more general than the model presented in equation (1.21), we write (1.21):

$$\frac{dS_t}{S_t} = |W_t| sgn(W_t) dB_t$$
$$S_0 = 1$$

Now we define $\beta_t = \int_0^t sgn(W_s)dB_s$, another Brownian motion which is independent of $(W_t, t \geq 0)$ and consequently of the reflecting Brownian motion $(|W_t|, t \geq 0)$. We get the following model:

$$\frac{dS_t}{S_t} = |W_t| d\beta_t, \qquad S_0 = 1$$

Now this modified form leads itself naturally to the generalization:

$$\frac{dS_t}{S_t} = R_t d\beta_t, \qquad S_0 = 1 \tag{1.22}$$

where, as in subsection 1.2.1, (R_t) denotes a Bessel process with dimension δ starting at 0 and (β_t) an independent Brownian motion.

Let us consider a Markovian martingale $(\Sigma_t, t \geq 0)$, which is the unique solution of:

$$\frac{d\Sigma_t}{\Sigma_t} = \sigma(t, \Sigma_t) d\beta_t \tag{1.23}$$
$$\Sigma_0 = 1$$

for some particular diffusion coefficient $\{\sigma(t, x), t \geq 0, x \in \mathbb{R}_+\}$ which has the same one-dimensional marginal distributions as $(S_t, t \geq 0)$ the solution of (1.22).

We will now use proposition 1.2.1 to find σ, the local volatility. We follow the notation in 1.2.1, and introduce a useful notation:

$$L_t^{(\mu)} = I_t - \mu A_t \tag{1.24}$$
$$\stackrel{(Law)}{=} N\sqrt{A_t} - \mu A_t \tag{1.25}$$

where N is a standard gaussian variable independent of A_t. Next we remark as a consequence of (1.25) that for any fixed $t \geq 0$:

$$(R_t, L_t^{(\mu)}) \stackrel{(Law)}{=} (R_t, N\sqrt{A_t} - \mu A_t)$$

and

$$\mathbb{E}\big[R_t^2|L_t^{(\mu)}=l\big]=\mathbb{E}\big[\mathbb{E}(R_t^2|N,A_t)|N\sqrt{A_t}-\mu A_t=l\big]$$

Since N is independent of R_t, we obtain

$$\mathbb{E}\big[R_t^2|L_t^{(\mu)}=l\big]=\mathbb{E}\big[\mathbb{E}(R_t^2|A_t)|N\sqrt{A_t}-\mu A_t=l\big]$$

From (1.3), we deduce:

$$\mathbb{E}\big[R_t^2|L_t^{(\mu)}=l\big]=\big(\frac{2}{t}\big)\mathbb{E}\big[A_t|N\sqrt{A_t}-\mu A_t=l\big] \tag{1.26}$$

Now, the computation of the expression in (1.26) is a simple exercise, which we present in the following form:

Lemma 1.4.1 *Let $X>0$ be a random variable independent from a standard gaussian variable N. Denote $Y^{(\mu)}=N\sqrt{X}-\mu X$. Then:*
i) for any $f:\mathbb{R}_+\to\mathbb{R}_+$, Borel function, the following formula holds:

$$\mathbb{E}\big[f(X)|Y^{(\mu)}=z\big]=\frac{h^{(\mu)}(f;z)}{h^{(\mu)}(1;z)}$$

where: $h^{(\mu)}(f;z)=\mathbb{E}\left[\frac{f(X)}{\sqrt{X}}\exp\left(-\frac{(z+\mu X)^2}{2X}\right)\right]$
ii) in particular, for $f(x)=x$, one can write :

$$\mathbb{E}\big[X|Y^{(\mu)}=z\big]=-\left(\frac{\frac{\partial k}{\partial b}}{k}\right)\big(\frac{z^2}{2},\frac{\mu^2}{2}\big) \tag{1.27}$$

where $k(a,b)=\mathbb{E}\left[\frac{1}{\sqrt{X}}\exp\left(-\big(\frac{a}{X}+bX\big)\right)\right]$

The proof of this lemma results from elementary properties of conditioning and is left to the reader.
We now give a formula for $\sigma^2(t,x)$ in terms of the law of $A_t\equiv A_t^{(\delta)}$, by using equation (1.26) and the above lemma. Indeed, it follows from these results that:

$$\mathbb{E}\big[R_t^2|\ln(S_t)=l\big]=-\frac{2}{t}\frac{\frac{\partial k_\delta^t}{\partial b}\big(\frac{l^2}{2},\frac{1}{8}\big)}{k_\delta^t\big(\frac{l^2}{2},\frac{1}{8}\big)}$$

where $k_\delta^t(a,b)=\mathbb{E}\left[\frac{1}{\sqrt{A_t}}\exp\left(-\big(\frac{a}{A_t}+bA_t\big)\right)\right]$.

Using the scaling property, we have $k_\delta^t(a,b)=\frac{1}{t}k_\delta^1(\frac{a}{t^2},bt^2)$ which allows us to concentrate on $k^\delta(a,b)\equiv k_\delta^1(a,b)$.
The following formula for the density f_δ of A_1 is borrowed from Biane, Pitman and Yor (2001). Denoting $h=\frac{\delta}{2}$, we have:

$$f_\delta(x)\equiv f_h^\sharp(x)=\frac{2^h}{\Gamma(h)}\sum_{n=0}^{\infty}(-1)^n\frac{\Gamma(n+h)}{\Gamma(n+1)}\frac{(2n+h)}{\sqrt{2\pi x^3}}\exp\left(-\frac{(2n+h)^2}{2x}\right) \tag{1.28}$$

For $\delta = 2$, $A^{(2)}$, or equivalently $f_2(x) = f_1^\sharp(x)$ enjoys a symmetry property (also shown in Biane, Pitman and Yor (2001)):
For any non-negative measurable function g

$$\mathbb{E}\Big[g(\frac{4}{\pi^2 A^{(2)}})\Big] = \sqrt{\frac{2}{\pi}}\mathbb{E}\Big[\frac{1}{\sqrt{A^{(2)}}}g(A^{(2)})\Big], \tag{1.29}$$

$$f_1^\sharp(x) = (\frac{2}{\pi x})^{\frac{3}{2}} f_1^\sharp(\frac{4}{\pi^2 x}) \tag{1.30}$$

and

$$f_1^\sharp(x) = \pi \sum_{n=0}^{\infty} (-1)^n (n + \frac{1}{2}) e^{-(n+\frac{1}{2})^2 \pi^2 \frac{x}{2}} \tag{1.31}$$

From formula (1.28), one may compute with the change of variables $a = \frac{\alpha^2}{2}, b = \frac{\beta^2}{2}$

$$\begin{aligned} k^\delta(a,b) &\equiv \mathbb{E}\Big[\frac{1}{\sqrt{A^{(\delta)}}}\exp\Big(-\frac{1}{2}(\frac{\alpha^2}{A^{(\delta)}} + \beta^2 A^{(\delta)})\Big)\Big] \\ &= \frac{2^h}{\Gamma(h)}\sum_{n=0}^{\infty}(-1)^n\frac{\Gamma(n+h)}{\Gamma(n+1)}\frac{2n+h}{\sqrt{2\pi}}\int_0^\infty \frac{dx}{x^2}e^{-\frac{1}{2}\left(\frac{\alpha^2+(2n+h)^2}{x}+\beta^2 x\right)} \\ &= \frac{2^h}{\Gamma(h)}\sum_{n=0}^{\infty}(-1)^n\frac{\Gamma(n+h)}{\Gamma(n+1)}\frac{2n+h}{\sqrt{2\pi}}\int_0^\infty e^{-\frac{1}{2}\left((\alpha^2+(2n+h)^2)x+\frac{\beta^2}{x}\right)}dx \qquad (\star) \end{aligned}$$

Also of importance for us, is the result:

$$\frac{\partial}{\partial b}(k^\delta(a,b)) = \frac{-(2^h)}{\Gamma(h)}\sum_{n=0}^{\infty}(-1)^n\frac{\Gamma(n+h)}{\Gamma(n+1)}\frac{2n+h}{\sqrt{2\pi}}\int_0^\infty \frac{e^{-\frac{1}{2}\left(\alpha_n^2 x+\frac{\beta^2}{x}\right)}}{x}dx \tag{1.32}$$

where $\alpha_n = \sqrt{\alpha^2 + (2n+h)^2}$.
Recall the integral representation for the Mc Donald functions K_ν:

$$K_\nu(z) \equiv K_{-\nu}(z) = \frac{1}{2}\big(\frac{z}{2}\big)^\nu \int_0^\infty \frac{dt}{t^{\nu+1}}\exp -\big(t + \frac{z^2}{2t}\big) \tag{1.33}$$

In particular, we have:

$$K_0(z) = \frac{1}{2}\int_0^\infty \frac{dt}{t}e^{-\left(t+\frac{z^2}{2t}\right)}$$

As a consequence:

$$\int_0^\infty \frac{du}{u}e^{-\frac{1}{2}\left(\alpha^2 u+\frac{\beta^2}{u}\right)} = 2K_0\Big(\frac{\alpha\beta}{\sqrt{2}}\Big) \tag{1.34}$$

Now, plugging (1.34) in (1.32), we obtain:

$$\frac{\partial}{\partial b}(k^\delta(a,b)) = \frac{-(2^h)}{\Gamma(h)}\sum_{n=0}^{\infty}(-1)^n\frac{\Gamma(n+h)}{\Gamma(n+1)}\frac{2n+h}{\sqrt{2\pi}}2K_0\Big(\frac{\alpha_n\beta}{\sqrt{2}}\Big) \tag{1.35}$$

Likewise, we deduce from (1.33) that:

$$K_1(z) \equiv K_{-1}(z) = \frac{1}{z}\int_0^{\infty} dt e^{-\left(t+\frac{z^2}{2t}\right)}$$

which implies

$$\int_0^{\infty} du e^{-\frac{1}{2}\left(\alpha_n^2 u+\frac{\beta^2}{u}\right)} = \frac{\beta\sqrt{2}}{\alpha_n}K_1(\frac{\alpha_n\beta}{\sqrt{2}})$$

Hence, we get as a consequence of ($\star$):

$$k^{\delta}(a,b) = \frac{2^h}{\Gamma(h)}\sum_{n=0}^{\infty}(-1)^n\frac{\Gamma(n+h)}{\Gamma(n+1)}\frac{2n+h}{\sqrt{\pi}}\frac{\beta}{\alpha_n}K_1\left(\frac{\alpha_n\beta}{\sqrt{2}}\right) \tag{1.36}$$

Recalling that $\beta = \sqrt{2b}$ and that $\alpha_n = \sqrt{2a+(2n+h)^2}$, we may now write (1.36) and (1.35) as:

$$k^{\delta}(a,b) = \frac{2^h}{\Gamma(h)}\sum_{n=0}^{\infty}(-1)^n\frac{\Gamma(n+h)}{\Gamma(n+1)}\frac{2n+h}{\sqrt{\pi}}\frac{\sqrt{2b}}{\alpha_n}K_1(\alpha_n\sqrt{b}) \tag{1.37}$$

$$\frac{\partial}{\partial b}(k^{\delta}(a,b)) = \frac{-(2^h)}{\Gamma(h)}\sum_{n=0}^{\infty}(-1)^n\frac{\Gamma(n+h)}{\Gamma(n+1)}\frac{2n+h}{\sqrt{2\pi}}2K_0(\alpha_n\sqrt{b}) \tag{1.38}$$

And finally, we obtain the following formula for the local volatility:

$$\sigma^2(t,x=e^l) = -\frac{4}{t}\left(\frac{\frac{\partial k^{\delta}}{\partial b}}{k^{\delta}}\right)(\frac{l^2}{2t^2},\frac{t^2}{8}) \tag{1.39}$$

1.4.2 Adding the Correlation

We now assume a non-zero correlation between the volatility process and the stock price process. This is a common fact in finance called the Leverage Effect and translated by a negative correlation. For a financial understanding of this effect, one can refer for instance to Black (1976), Christie (1982) or Schwert (1989).

Let us define our new model for the stock price dynamics with a Bessel process of dimension δ starting from 0 correlated to the Brownian motion of the stock price process:

$$\frac{dS_t}{S_t} = R_t dW_t \tag{1.40}$$

$$dR_t^2 = 2R_t dW_t^{\sigma} + \delta t \tag{1.41}$$

$$d < W^{\sigma}, W >_t = \rho dt \tag{1.42}$$

$$S_0 = 1 \qquad and \qquad R_0 = 0 \tag{1.43}$$

Then, there exists a Brownian motion β independent of the Bessel process such that $\forall t$:

$$W_t = \rho W_t^{\sigma} + \sqrt{1-\rho^2}\beta_t$$

Using the previous formula, plugging it in (1.40) and then inserting (1.41) in the new (1.40), one gets:

$$\frac{dS_t}{S_t} = \frac{\rho}{2}(dR_t^2 - \delta dt) + \sqrt{1-\rho^2}R_t d\beta_t \tag{1.44}$$

Then using Itô formula applied to $f(x) = ln(x)$

$$d\ln(S_t) = \frac{dS_t}{S_t} - \frac{1}{2}R_t^2 dt$$

one obtains:

$$\ln(S_t) = \frac{\rho}{2}(R_t^2 - \delta t) + \sqrt{1-\rho^2}\int_0^t R_s d\beta_s - \frac{1}{2}\int_0^t R_s^2 ds \tag{1.45}$$

Let us consider as in subsection 1.4.1, $L_t^{(\mu)} = \int_0^t R_s d\beta_s - \mu \int_0^t R_s^2 ds$ (we are especially interested in the case $\mu = \frac{1}{2\sqrt{1-\rho^2}}$). Since R and β are independent, we shall use the same notation as above. Particularly, A_t and I_t will refer to the quantities defined in subsection 1.2.1
Now, equation (1.45) can be rewritten as follows:

$$\ln(S_t) = \frac{\rho}{2}(R_t^2 - \delta t) + \sqrt{1-\rho^2} L_t^{(\frac{1}{2\sqrt{1-\rho^2}})} \tag{1.46}$$

Since we wish to evaluate the local volatility $\mathbb{E}\big[R_t^2 | \ln(S_t) = l\big]$, we will try to compute more generally the following quantity:

$$\mathbb{E}\big[R_t^2 | mR_t^2 + L_t^{(\mu)} = l\big] \tag{1.47}$$

where m is a real constant.

Remark 1.4.2 *We immediately see that if we take $m = 0$, i.e $\rho = 0$, we are back to the previous paragraph setting.*

First, we see that equation (1.26) is easily extended to the case with correlation and we obtain:

$$\mathbb{E}\big[R_t^2 | mR_t^2 + L_t^{(\mu)} = l\big] = \frac{2}{t}\mathbb{E}\big[A_t | mR_t^2 + L_t^{(\mu)} = l\big] \tag{1.48}$$

Before extending Lemma 1.4.1, one must recall that for any $t \geq 0$:

$$(R_t, A_t, L_t^{(\mu)}) \overset{(Law)}{=} (R_t, A_t, N\sqrt{A_t} - \mu A_t)$$

where N is a standard gaussian variable independent of R_t and A_t. The following simple result will be helpful for the remaining of the paper:

Lemma 1.4.3 *Let $X > 0$ and $Z \geq 0$ independent from a standard gaussian variable N. Denote $Y^{(\mu)} = N\sqrt{X} - \mu X$. Then:*
i) for any Borel function $f : \mathbb{R}_+^2 \to \mathbb{R}_+$, real number m we have the formula:

$$\mathbb{E}\big[f(X,Z) | mZ + Y^{(\mu)} = z\big] = \frac{a^{(\mu,m)}(f;z)}{a^{(\mu,m)}(1;z)}$$

where: $a^{(\mu,m)}(f;z) = \mathbb{E}\left[\frac{f(X,Z)}{\sqrt{X}} \exp\left(-\frac{(z+\mu X - mZ)^2}{2X}\right)\right]$
ii) in particular, for $f(x,y) = x$, we obtain:

$$\mathbb{E}\big[X | mZ + Y^{(\mu)} = z\big] = -\frac{1}{\mu\sqrt{2}}\left(\frac{\frac{\partial\alpha}{\partial b}}{\alpha}\right)\left(\frac{z}{\sqrt{2}}, \frac{\mu}{\sqrt{2}}, \frac{m}{\sqrt{2}}\right) \tag{1.49}$$

where $\alpha(a,b,c) = \mathbb{E}\Bigg[\frac{1}{\sqrt{X}}\exp\Bigg(-\big(\frac{(a-cZ)^2}{X}+b^2X+bcZ\big)\Bigg)\Bigg]$

The other fundamental result we now need, is the joint density of $(R_t^2, \int_0^t R_s^2 ds)_{t\geq 0}$.

Theorem 1.4.4 *The joint density g_t of $(R_t^2, \int_0^t R_s^2 ds)$ is given by:*

$$g_t(x,y) = \frac{1}{\sqrt{2\pi}\Gamma(\frac{\delta}{2})}\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}x^{j+\frac{\delta}{2}-1}y^{-\frac{j}{2}-\frac{\delta}{4}-1}f_t^j(x,y) \tag{1.50}$$

where f_t^j is defined by

$$f_t^j(x,y) = \sum_{k=0}^{\infty}\frac{(j+\frac{\delta}{2})_k}{k!}e^{-\frac{1}{4y}[2(k+j+\frac{\delta}{4})t+\frac{x}{2}]^2}D_{\frac{\delta}{2}+j+1}\big(\frac{2(k+j+\frac{\delta}{4})t+\frac{x}{2}}{\sqrt{y}}\big) \tag{1.51}$$

$D_\nu(\xi)$ is a parabolic cylinder function and $(\nu)_k$ the Pochhammer's symbol defined by $(\nu)_k \equiv \nu(\nu+1)...(\nu+k-1) = \Gamma(\nu+k)/\Gamma(\nu)$

Proof. See Ghomrasni (2004) who evaluates the Laplace transform of (1.2) in order to get the density function. ∎

For the definition and properties on the parabolic cylinder functions, we refer to Gradshteyn and Ryzhik (2000).
Let us define α_t in the following form:

$$\alpha_t(a,b,c) = \mathbb{E}\Bigg[\frac{1}{\sqrt{A_t}}\exp\Bigg(-\big(\frac{(a-cR_t^2)^2}{A_t}+b^2A_t+bcR_t^2\big)\Bigg)\Bigg] \tag{1.52}$$

Unfortunately, there is no more scaling property as in the zero-correlation case and we may not rewrite α_t as a function of t and α_1. One can then compute the local volatility $\sigma^{(\rho)}$ by noticing that in the case of particular interest for us, the parameters are defined as follows:

$$m = \frac{\rho}{2\sqrt{1-\rho^2}} \qquad \textit{and} \qquad z = \frac{l+\frac{\rho}{2}\delta t}{\sqrt{1-\rho^2}} \qquad \textit{and} \qquad \mu = \frac{1}{2\sqrt{1-\rho^2}}$$

We then obtain

$$\sigma^{(\rho)}(t, x=e^l) = -\sqrt{2(1-\rho^2)}\left(\frac{\frac{\partial\alpha_t}{\partial b}}{\alpha_t}\right)\big(\frac{l+\frac{\rho}{2}\delta t}{\sqrt{2(1-\rho^2)}}, \frac{1}{\sqrt{8(1-\rho^2)}}, \frac{\rho}{\sqrt{8(1-\rho^2)}}\big) \tag{1.53}$$

1.4.3 From a Bessel Volatility process to the Heston Model

The Heston (1993) model for representing a stochastic volatility process is a particular case of the Cox, Ingersoll and Ross (1985) stochastic process, of the form:

$$dV_t = \kappa(\theta - V_t)dt + \eta\sqrt{V_t}dW_t \tag{1.54}$$

with initial condition $V_0 = v_0$

Actually, it is possible to find out deterministic space and time changes such as the law of the Heston SDE solution and the Time-Space transformed Bessel Process are the same.

Proposition 1.4.5 *For every Heston SDE solution, there exist a Bessel process and two deterministic functions f and g with g increasing such as:*

$$V_t = f(t) \times R^2_{g(t)}$$

where R denotes a Bessel Process of dimension $\delta = \frac{4\kappa\theta}{\eta^2}$ starting from $\sqrt{v_0}$ at time $t = 0$ and f and g are defined by:

$$f(t) = e^{-\kappa t}$$
$$g(t) = \frac{\eta^2}{4\kappa}(e^{\kappa t} - 1)$$

Proof. It is just an application of Lemma 1.2.4. ■

One may now apply the results of the previous sections using the time and space transformations presented in the previous paragraph

Proposition 1.4.6 *Let us consider the following stochastic volatility model:*

$$\begin{aligned} \frac{dS_t}{S_t} &= \sqrt{v_t} d\beta_t, \qquad S_{\{t=0\}} = S_0 \\ v_t &= \frac{\eta^2}{4} e^{2\kappa t} V_t, \qquad v_0 = \frac{\eta^2}{4} V_0 \\ dV_t &= \kappa(\theta - V_t)dt + \eta\sqrt{V_t} dW_t \\ d < \beta_., W_. >_t &= \rho dt \end{aligned}$$

where β_t is a Brownian motion and V_t is an Heston process as defined above.
Then the local volatility $\tilde{\sigma}$ that gives us the expected mimicking properties, satisfies the following equation:

$$\tilde{\sigma}(t, x) = \frac{\eta^2}{4} e^{\kappa t} \sigma\big(\frac{\eta^2}{4\kappa}(e^{\kappa t} - 1), \frac{x}{s_0}\big) \tag{1.55}$$

where $\sigma^2(t, x) = \mathbb{E}[R_t^2 | \exp(I_t - \frac{1}{2}A_t) = x]$ and R_t is a Bessel Process of dimension δ starting from V_0 if $\rho = 0$ and $\sigma^2(t, x) = (\sigma^{(\rho)}(t, x))^2$ as defined above otherwise.

Proof. First, one has the Gyöngy volatility formula:

$$\tilde{\sigma}^2(t, x) = \frac{\eta^2}{4} e^{2\kappa t} \mathbb{E}\big[V_t | S_t = x\big] \tag{1.56}$$

Then using Lemma 1.2.4, one easily obtains the result. ■

Remark 1.4.7 *Let us note that we only have closed-form formulas in cases where $V_0 = 0$ and that otherwise we have to go through Laplace transform inversion techniques.*

One can propose a general framework for constructing stochastic volatility models based on Bessel processes. Local volatilities can be computed through the proposition below whose proof is left to the reader.

Proposition 1.4.8 *Let us consider the following stochastic volatility model:*

$$\begin{aligned}
\frac{dS_t}{S_t} &= \sqrt{v_t}d\beta_t, \qquad S_{\{t=0\}} = S_0 \\
v_t &= \frac{g'(t)}{f(t)}V_t, \qquad v_0 = \frac{g'(0)}{f(0)}V_0 \\
dV_t &= \left(\delta f(t)g'(t) + \frac{f'(t)}{f(t)}V_t\right)dt + \sqrt{f(t)}g'(t)\sqrt{V_t}dW_t \\
d < \beta_., W_. >_t &= \rho dt
\end{aligned}$$

where β_t and W_t are Brownian motions, f is a positive continuously differentiable function and g an increasing $\mathcal{C}^1$ function.
Then the local volatility $\tilde{\sigma}$ that gives us the expected mimicking properties, satisfies the following equation:

$$\tilde{\sigma}(t,x) = g'(t)\sigma\big(g(t), \frac{x}{S_0}\big) \tag{1.57}$$

where $\sigma^2(t,x) = \mathbb{E}[R_t^2| \exp(I_t - \frac{1}{2}A_t) = x]$ and R_t is a Bessel Process of dimension δ starting from V_0 if $\rho = 0$ and $\sigma^2(t,x) = (\sigma^{(\rho)}(t,x))^2$ as defined above otherwise.

1.5 Pricing Equity Derivatives under Stochastic Interest Rates

1.5.1 A Local Volatility Framework

With the growth of hybrid products, it has been necessary to take properly into account the stochasticity of interest rates in FX or Equity models in a way that makes the equity volatility surface calibration easy at a given interest rate parametrization. It has been now a while that people have been considering interest rates as stochastic for long-dated Equity or FX options, but they have not been thinking about it in terms of calibration issues. Besides, according to the interest rates part of an equity - interest rates hybrid product for example, the instruments on which the interest rates model will be calibrated are different; hence it becomes necessary to parameterize the volatility surface efficiently. For most of hybrid products, no forward volatility dependence is involved and then a local volatility framework is sufficient. Let us now consider a local volatility model with stochastic interest rates:

$$\frac{dS_t}{S_t} = r_t dt + \sigma(t, S_t)dW_t$$

where r_t is a stochastic process and σ a deterministic function.
Now, we can observe that equation (1.18) is still valid under stochastic rates and we may then write

$$d_tC(t,x) = x\mathbb{E}[r_t e^{-\int_0^t r_s ds}\mathbf{1}_{\{S_t > x\}}]dt + \frac{1}{2}\lim_{\epsilon\to 0}\mathbb{E}[\frac{1}{\epsilon}\mathbf{1}_{\{x\leq S_t < x+\epsilon\}}e^{-\int_0^t r_s ds}\sigma^2(t,S_t)S_t^2]dt$$

The second term of the right-hand side may be written as follows

$$\mathbb{E}[e^{-\int_0^t r_s ds}\sigma^2(t,S_t)S_t^2\mathbf{1}_{\{x\leq S_t<x+\epsilon\}}] = \mathbb{E}\big[\mathbb{E}[e^{-\int_0^t r_s ds}|S_t]\sigma^2(t,S_t)S_t^2\mathbf{1}_{\{x\leq S_t<x+\epsilon\}}\big]$$

and then we have

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\mathbb{E}[e^{-\int_0^t r_s ds}\sigma^2(t,S_t)S_t^2\mathbf{1}_{\{x\leq S_t<x+\epsilon\}}] = x^2\sigma^2(t,x)q_t(x)\mathbb{E}[e^{-\int_0^t r_s ds}|S_t = x]$$

where $q_t(x)$ is the value of the density of S_t in x. It is easily shown as well that

$$\frac{\partial^2 C}{\partial x^2} = q_t(x)\mathbb{E}[e^{-\int_0^t r_s ds}|S_t = x]$$

Let us now define the t-forward measure $\mathbb{Q}^t$ (see Geman (1989), Jamshidian (1989)) by

$$\frac{d\mathbb{Q}^t}{d\mathbb{Q}} = \frac{e^{-\int_0^t r_s ds}}{B(0,t)} \qquad \text{where} \qquad B(0,t) = \mathbb{E}[e^{-\int_0^t r_s ds}]$$

Hence, we finally obtain an extension of Dupire (1994)'s formula :

$$\sigma^2(t,x) = \frac{\frac{\partial C}{\partial t} - xB(0,t)\mathbb{E}^t[r_t \mathbf{1}_{\{S_t > x\}}]}{\frac{x^2}{2}\frac{\partial^2 C}{\partial x^2}}$$

Under a T-forward measure for $T \geq t$, one has

$$\mathbb{E}^T[r_T|\mathcal{F}_t] = f(t,T)$$

where $f(t,T)$ is the instantaneous forward rate. To conclude this subsection, we can first notice that this slight extension of Dupire equation may be also written

$$\sigma^2(t,x) = \frac{\frac{\partial C}{\partial t} + xf(0,t)\frac{\partial C}{\partial x} - xB(0,t)\mathbb{C}ov^t(r_t; \mathbf{1}_{\{S_t > x\}})}{\frac{x^2}{2}\frac{\partial^2 C}{\partial x^2}} \tag{1.58}$$

We then assume that it is possible to extract from markets prices the quantities $\mathbb{C}ov^t(r_t; \mathbf{1}_{\{S_t > x\}})$ (i.e. there exist tradeable assets from which we could obtain these covariances) in order to add stochastic interest rates to the usual local volatility framework. For the remainder of the paper, we denote this assumption the (HC)-Hypothesis that stands for Hybrid Correlation hypothesis. Under this market hypothesis, one is able to calibrate a local volatility surface with stochastic interest rates implied by the derivatives' market prices.

1.5.2 Mimicking Stochastic Volatility Models

In this subsection, we consider the case of a stochastic volatility model with stochastic interest rates and see how it is possible to connect it to a local volatility framework. Let us consider the following diffusion

$$\frac{dS_t}{S_t} = r_t dt + \sqrt{V_t}dW_t$$

with V_t a stochastic process and let us use equation (1.18) in order to exhibit a new mimicking property:

$$d_t C(t,x) = x\mathbb{E}[r_t e^{-\int_0^t r_s ds}\mathbf{1}_{\{S_t > x\}}]dt + \frac{1}{2}\lim_{\epsilon \to 0}\mathbb{E}[\frac{1}{\epsilon}\mathbf{1}_{\{x \leq S_t < x+\epsilon\}}e^{-\int_0^t r_s ds}V_t S_t^2]dt$$

Then,

$$\begin{aligned}
\lim_{\epsilon \to 0}\frac{1}{\epsilon}\mathbb{E}[e^{-\int_0^t r_s ds}V_t S_t^2 \mathbf{1}_{\{x \leq S_t < x+\epsilon\}}] &= \lim_{\epsilon \to 0}\frac{1}{\epsilon}\mathbb{E}\left[\mathbb{E}[V_t e^{-\int_0^t r_s ds}|S_t]S_t^2 \mathbf{1}_{\{x \leq S_t < x+\epsilon\}}\right] \\
&= x^2 q_t(x)\mathbb{E}[V_t e^{-\int_0^t r_s ds}|S_t = x] \\
&= x^2 \frac{\partial^2 C}{\partial x^2}\frac{\mathbb{E}[V_t e^{-\int_0^t r_s ds}|S_t = x]}{\mathbb{E}[e^{-\int_0^t r_s ds}|S_t = x]}
\end{aligned}$$

Hence, we obtain

$$\frac{\mathbb{E}[V_t e^{-\int_0^t r_s ds}|S_t = x]}{\mathbb{E}[e^{-\int_0^t r_s ds}|S_t = x]} = \frac{\frac{\partial C}{\partial t} + xf(0,t)\frac{\partial C}{\partial x} - xB(0,t)\mathbb{C}ov^t(r_t; \mathbf{1}_{\{S_t > x\}})}{\frac{x^2}{2}\frac{\partial^2 C}{\partial x^2}}$$

Spot Mimicking Property Finally, if there exists a stochastic process, solution of the following SDE

$$\frac{dX_t}{X_t} = r_t dt + \sigma(t, X_t) dW_t$$

such that the one-dimensional marginals of the triple $(r_t, \int_0^t r_s ds, X_t)$ are the same as $(r_t, \int_0^t r_s ds, S_t)$, then by identification one must have

$$\sigma^2(t,x) = \frac{\mathbb{E}[V_t e^{-\int_0^t r_s ds}|S_t = x]}{\mathbb{E}[e^{-\int_0^t r_s ds}|S_t = x]}$$

The existence is easily proven in the cases where $(r_t, t \geq 0)$ is a Markovian diffusion. Hence, we exhibit a strong mimicking property since we obtained an explicit way to construct a local volatility surface.

Remark 1.5.1 *We may notice that if interest rates are deterministic, we recover the well-known formula (1.20).*

Forward Mimicking Property Let us now write a Forward mimicking property by applying Gyöngy's result to match the one dimensional marginals of a stochastic volatility model and of a local volatility one:
If one defines $F_t^{(1)} = S_t e^{-\int_0^t r_s ds}$ and $F_t^{(2)} = X_t e^{-\int_0^t r_s ds}$ where S and X are defined above, we obtain the existence of diffusions $Y_t^{(1)}$ and $Y_t^{(2)}$ solutions of

$$\frac{dY_t^{(i)}}{Y_t^{(i)}} = \Sigma_{(i)}(t, Y_t^{(i)}) dW_t$$

for $i = 1, 2$ such as

$$\begin{aligned} \Sigma^2_{(1)}(t,x) &= \mathbb{E}[V_t | S_t = x e^{\int_0^t r_s ds}] \\ \Sigma^2_{(2)}(t,x) &= \mathbb{E}[\sigma^2(t, x e^{-\int_0^t r_s ds}) | X_t = x e^{\int_0^t r_s ds}] \end{aligned}$$

Since the one-dimensional marginals of $F_t^{(1)}$ and $F_t^{(2)}$ must be equal, one obtains

$$\mathbb{E}[V_t | S_t = x e^{\int_0^t r_s ds}] = \mathbb{E}[\sigma^2(t, x e^{-\int_0^t r_s ds}) | X_t = x e^{\int_0^t r_s ds}] \tag{1.59}$$

We consequently obtain an implicit way to construct a local volatility surface we say that this relation is weak in the sense that it is a weak mimicking distribution property which is involved in the above relation.

1.5.3 From a Deterministic Interest Rates Framework to a Stochastic one

Going from a framework to another is valuable for calibration issues. Let us assume, for instance that a model has been calibrated with deterministic interest rates and that one wants to recalibrate the same model assuming stochastic interest rates. Let us introduce some notation to define the different kinds of frameworks we will go through in this subsection.

Notation
LV stands for Local Volatility, **SV** stands for Stochastic Volatility, **DIR** stands for Deterministic Interest Rates and **SIR** stands for Stochastic Interest Rates

From DIR-LV to SIR-LV Let us first consider the local volatility case. Under deterministic interest rates, the stock price dynamics are driven by the equation

$$\frac{dS_t}{S_t} = r(t)dt + \sigma(t, S_t)dW_t$$

while under stochastic interest rates it would be

$$\frac{dS_t}{S_t} = r_t dt + \overline{\sigma}(t, S_t)dW_t$$

and we know that both local volatility functions solve the following implied equations:

$$\begin{aligned}\sigma^2(t,x) &= \frac{\frac{\partial C}{\partial t} + xf(0,t)\frac{\partial C}{\partial x}}{\frac{x^2}{2}\frac{\partial^2 C}{\partial x^2}} \\ \overline{\sigma}^2(t,x) &= \frac{\frac{\partial C}{\partial t} + xf(0,t)\frac{\partial C}{\partial x} - xB(0,t)\mathbb{C}ov^t(r_t;\mathbf{1}_{\{S_t>x\}})}{\frac{x^2}{2}\frac{\partial^2 C}{\partial x^2}}\end{aligned}$$

where $f(0,t) = r(t)$.

Now, if the prices involved in the estimation of the local volatility surfaces are observed on markets and respect the (HC)-Hypothesis, one may write

$$\sigma^2(t,x) - \overline{\sigma}^2(t,x) = \frac{2B(0,t)\mathbb{C}ov^t(r_t;\mathbf{1}_{\{S_t>x\}})}{x\frac{\partial^2 C}{\partial x^2}} \tag{1.60}$$

From DIR-SV to SIR-SV If we assume that a general Itô process drives the volatility we will write

$$\begin{aligned}\frac{dS_t^{(1)}}{S_t^{(1)}} &= r(t)dt + \sqrt{V_t^{(1)}}dW_t \\ \frac{dS_t^{(2)}}{S_t^{(2)}} &= r_t dt + \sqrt{V_t^{(2)}}dW_t\end{aligned}$$

and then, if $S^{(1)}$ and $S^{(2)}$ have the same one-dimensional marginals, we obtain the following relation to relate $V^{(1)}$ to $V^{(2)}$:

$$\mathbb{E}[V_t^{(1)}|S_t^{(1)}=x]-\frac{\mathbb{E}[V_t^{(2)}e^{-\int_0^t r_s ds}|S_t^{(2)}=x]}{\mathbb{E}[e^{-\int_0^t r_s ds}|S_t^{(2)}=x]}=\frac{2B(0,t)\mathbb{C}ov^t(r_t;\mathbf{1}_{\{S_t^{(2)}>x\}})}{x\frac{\partial^2 C}{\partial x^2}} \tag{1.61}$$

From SIR-SV to DIR-LV Let us now specify a Heath Jarrow and Morton (1992) diffusion for the interest rate model and see precisely how one could extract, using Gyöngy's result, the volatility of the forward contract under deterministic interest rates from the volatility of the forward contract under stochastic interest rates. Let us recall that in a standard HJM framework, the instantaneous forward rate follows

$$df(t,T)=\Big(\sigma(t,T)\int_t^T\sigma(t,u)du\Big)dt+\sigma(t,T)dW_t^r$$

where $\sigma(t,T)$ is a stochastic process adapted to its canonical filtration and where the price satisfies

$$B(t,T)=\exp\bigg(-\int_t^T f(t,s)ds\bigg)$$

By definition $r_t=f(t,t)$ and then we obtain

$$\begin{aligned}\frac{dB(t,T)}{B(t,T)} &= r_t dt-\sigma_B(t,T)dW_t^r\\ \sigma_B(t,T) &= \int_t^T\sigma(t,u)du\end{aligned}$$

For our purpose, let us consider a general model

$$\frac{dS_t}{S_t}=r_t dt+\sqrt{V_t}dW_t$$

then recall the price of the T-forward contract written on S

$$F_t^T=\frac{S_t}{B(t,T)}$$

where we assume $d<W,W^r>_t=\rho dt$. We are now able to write the dynamics of F_t^T under $\mathbb{Q}$ the risk-neutral measure:

$$\frac{dF_t^T}{F_t^T}=\frac{dS_t}{S_t}-\frac{d<S.,B(\cdot,T)>_t}{S_tB(t,T)}-\bigg(\frac{dB(t,T)}{B(t,T)}-\frac{d<B(\cdot,T)>_t}{B^2(t,T)}\bigg)$$

If we introduce the T-forward probability measure as above by

$$\frac{d\mathbb{Q}^T}{d\mathbb{Q}}=\frac{e^{-\int_0^T r_s ds}}{B(0,T)}$$

we explain the dynamics of F_t^T under $\mathbb{Q}^T$

$$\frac{dF_t^T}{F_t^T}=\sqrt{V_t}d\widetilde{W}_t+\sigma_B(t,T)d\widetilde{W}_t^r$$

where $\widetilde{W}$ and $\widetilde{W}^r$ are Brownian motions under $\mathbb{Q}^T$ such as $d < \widetilde{W}, \widetilde{W}^r >_t = \rho dt$.
We now apply Theorem 1.2.11 and obtain the existence of a process $\widetilde{F}_t^T$ solution of an inhomogeneous Markovian stochastic differential equation

$$\frac{d\widetilde{F}_t^T}{\widetilde{F}_t^T} = \Sigma_T(t, \widetilde{F}_t^T) d\beta_t$$

where β is a Brownian motion and

$$\Sigma_T^2(t,x) = \mathbb{E}^T[V_t + 2\rho\sqrt{V_t}\sigma_B(t,T) + \sigma_B^2(t,T)|S_t = xB(t,T)]$$

If we consider a local volatility model with deterministic interest rates as follows

$$\frac{dS_t}{S_t} = f(0,t)dt + \sigma(t, S_t)d\gamma_t$$

the dynamics of the T-forward contract then becomes

$$\frac{dF_t^T}{F_t^T} = \sigma(t, F_t^T e^{-\int_t^T f(0,s)ds})d\gamma_t$$

and Gyöngy's result enables us to conclude that

$$\Sigma_T(t,x) = \sigma(t, xe^{-\int_t^T f(0,s)ds})$$

Hence, we have proven a new relation that links a local volatility framework with deterministic interest rates to a stochastic volatility one with stochastic interest rates, namely

$$\sigma^2(t,x) = \mathbb{E}^T[V_t + 2\rho\sqrt{V_t}\sigma_B(t,T) + \sigma_B^2(t,T)|S_t = xB(t,T)e^{\int_t^T f(0,s)ds}] \tag{1.62}$$

An illustration of this formula can be found for a Black and Scholes (1973) framework with random rates for example in Hull and White (1994).

1.6 Conclusion

This paper recalls well-known results on local volatility and establishes links to stochastic volatility through the powerful theorems of Krylov and Gyöngy. These general results are then illustrated with explicit computations of local volatility in different stochastic volatility models where the volatility process is a time-space transformation of Bessel processes. In this framework, we show the impact of the stock-volatility correlation on the local volatility surface.
The local volatility extracted from a stochastic volatility model allows us to get a precise idea of the skew generated by a stochastic volatility model. Hence, an important theoretical and numerical advantage of generating a local volatility surface from a stochastic volatility rather than from market option prices is the stability and the meaningfulness of the surface. Indeed, the local volatility surface constructed with the Forward PDE equation is known to be completely unstable whereas as one can see the one built from a stochastic volatility is really smooth.
With the growth of hybrid products, it has been important to seriously consider the issue of volatility calibration under stochastic interest rates and that is the reason why we exhibit different relations between local volatilities, stochastic volatilities and derivative prices. It is shown that Dupire (1994) and Derman and Kani (1998) formulas can easily be extended and that it is possible to relate any continuous stochastic volatility model with stochastic interest rates to a local volatility one with deterministic interest rates. By extending the local volatility formula to a stochastic rates framework, it is observed that a market premium for the hybrid correlation risk is to be implied for the construction of the local volatility surface, which can be performed under the (HC)-Hypothesis as at some point a market premium for the volatility risk is to be taken into account.
A remaining interesting question is the existence of a local volatility diffusion with a general Ito interest rates process framework such that the joint law of instantaneous rate, the discount factor and the stock price is the same as the one in a stochastic volatility framework.

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Chapter 2

Time-Changed Bessel Processes and Credit Risk

[*Joint work with Boris Leblanc; submitted for publication**]

The Constant Elasticity of Variance (CEV) model is mathematically presented and then used in a Credit-Equity hybrid framework. Next, we propose extensions to the CEV model with default: firstly by adding a stochastic volatility diffusion uncorrelated from the stock price process, then by more generally time changing Bessel processes and finally by correlating stochastic volatility moves to the stock ones. Properties about strict local and true martingales in this study are discussed. Analytical formulas are provided and Fourier and Laplace transform techniques can then be used to compute option prices and probabilities of default.

2.1 Introduction

It has been widely recognized for at least a decade that the option pricing theory of Black and Scholes (1973) and Merton (1973) is not consistent with market option prices and underlying dynamics. It has been noted that options with different strikes and maturities have different implied volatilities. Indeed, markets take into account in option prices the presence of skewness and kurtosis in the probability distributions of log returns. In order to deal with those effects, one could use stochastic volatility models (e.g. Heston (1993), Hull and White (1987) or Scott (1987)). Another common alternative is to use a deterministic time and stock price dependent volatility function, the so-called local volatility to capture these effects. One would then build the volatility surface by excerpting the values of this function from option prices, thanks to the well-known Derman and Kani (1994) and Dupire (1994) formula.

One of the first models developed after Black Merton Scholes (1973) is the Constant Elasticity of Variance model pioneered by Cox (1975) where the volatility is a deterministic function of the spot level; This latter model is somehow an ancestor of local volatility models. It has very interesting features since it suggests that common stock returns are heteroscedastic and that volatilities implied by the Black and Scholes formula are not constant, in other words skew exists in this model. Another

*We thank Marc Yor for all the important contributions to this paper.

interesting property is that it takes into account the so called "Leverage Effect" which considers the effects of financial leverage on the variance of a stock: a stock price increase reduces the debt-equity ratio of a firm and therefore decreases the variance of the stock's returns (see for instance Black (1976), Christie (1982) or Schwert (1989)). A last but not least feature of this model is that it has a non-zero probability of hitting 0 and this could be of importance when one is interested in modelling default by defining bankruptcy as the stock price falling to 0.

For the last few years, the credit derivatives market has become more and more important and the issue of modeling default has grown, giving birth to two main classes of models. The first class is the structural models of the firm pioneered by Merton (1974) where bankruptcy occurs if the asset value falls to a boundary determined by outstanding liabilities. Other early work on such models was done by Black and Cox (1976) and Geske (1977). The other class commonly called reduced-form models is less ambitious than structural models. They consider the time of default as an exogenous parameter that they calibrate under a risk neutral probability to market data. These models were developed by Artzner and Delbaen (1995), Jarrow and Turnbull (1995), Duffie, Schroder and Skiadas (1996) and Madan and Unal (1998).

The credit risk is also a component of the equity derivatives market as it may appear in convertible bonds or more generally in Capital Structure Arbitrage for people that embedded it from out-of-the money puts. It is then clear that having a consistent modeling of equity and credit is essential to eventually be able to manage those cross-asset positions. Indeed, a market standard has been developed during the last few years which involves a jump diffusion dynamics for the stock price with a local probability of default for the jump factor. This kind of model has been presented for instance in Ayache, Forsyth and Vetzal (2003). An important drawback of this modeling is that the stock has to jump to zero in order to default, which isn't a realistic assumption as we can see on several historical data and as argued in Atlan and Leblanc (2005).

The necessity to have stock price diffusions that don't jump to zero in order to default and still have a non-zero probability of falling to zero leads us to naturally consider CEV processes. Moreover, CEV models have the advantage to provide closed-form formulas for European vanilla options and for the probability of default. Those computations were originally performed by Cox (1975) in the case where the stock can default and by Emanuel and McBeth (1982) when the stock never defaults. Then, one may want to add a stochastic volatility process to the CEV diffusion in order to capture some volatility features such as a smile or such as a more realistic volatility term structure. Finally, to get more dependency between the stock price and the volatility, one may add some correlation.

Those guidelines lead us to study in section 2 the one-dimensional marginals, the first-passage times below boundaries and the default of martingality of Constant Elasticity of Variance processes, mainly by relating those latest to Bessel processes. In section 3, we propose a CEV model that is stopped at its default time and we provide closed form formulas for European vanilla options, Credit Default Swaps and Equity Default Swaps. Section 4 extends the Constant Elasticity of Variance framework to a Constant Elasticity of Stochastic Variance one by firstly adding a stochastic volatility to the CEV diffusion and in a second time more generally consider time-changed Bessel processes with a stochastic integrated time change. Quasi-analytical formulas conditionally on the knowledge of the law of the time change are provided for vanilla options and CDSs and examples are given. Section 5 adds a correlation term to the general time-changed power of Bessel process framework, once again quasi analytical formulas conditionally on the knowledge of the joint law of the time change and a process related to the rate of time change are provided for probabilities of default and for vanilla options, and computations for several examples are shown. All the models proposed in this paper are true martingales and the martingale property is carefully proven for the different frameworks. Finally, section 6 concludes and presents possible extensions of this work.

Convention *For strictly negative dimensions we define squared Bessel processes up to their first hitting time of* 0 *after which they remain at* 0.

We set this convention because we wish to consider positive Bessel processes. For a study of negative dimension Bessel processes with negative values, we refer to Göing-Jaeschke and Yor (2003).

2.2 A Mathematical Study of CEV Processes

2.2.1 Space and Time Transformations

A reason why Bessel processes play a large role in financial mathematics is that they are closely related to widely used models such as Cox, Ingersoll and Ross (1985), i.e. the CIR family of diffusions for interest rates framework, such as the Heston (1993) stochastic volatility model or even to the Constant Elasticity of Variance model of Cox (1976). They are more generally related to exponential of time-changed Brownian motions thanks to Lamperti (1972) representations.

Let us now concentrate on the CIR family of diffusions: they solve the following type of stochastic differential equations:

$$dX_t = (a - bX_t)dt + \sigma\sqrt{|X_t|}dW_t \tag{2.1}$$

with $X_0 = x_0 > 0$, $a \in \mathbb{R}_+$, $b \in \mathbb{R}$, $\sigma > 0$ and W_t a standard Brownian motion. This equation admits a strong (e.g. adapted to the natural filtration of W_t) unique solution that takes values in $\mathbb{R}_+$.

Let us remark that squared Bessel processes of dimension $\delta > 0$ can be seen as a particular case of a CIR process with $a = \delta$, $b = 0$ and $\sigma = 2$. We also recall that a Bessel process R_t solves the following diffusion equation

$$dR_t = dW_t + \frac{\delta - 1}{2R_t}dt$$

where for $\delta = 1$, the latter $\frac{\delta-1}{2R_t}dt$ must be replaced by a local time term.

One is now interested in the representation of a CIR process in terms of a time-space transformation of a Bessel Process:

Lemma 2.2.1 *A CIR Process X_t which solves equation (2.1) can be represented in the following form:*

$$X_t = e^{-bt}BESQ_{(\delta,x_0)}(\frac{\sigma^2}{4b}(e^{bt} - 1)) \tag{2.2}$$

where $BESQ_{(\delta,x_0)}$ denotes a squared Bessel Process starting from x_0 at time $t = 0$ of dimension $\delta = \frac{4a}{\sigma^2}$

Proof. This lemma results from the identification of two continuous functions f and g (with g strictly increasing and $g(0) = 0$) such as

$$X_t = f(t)BESQ_{(\delta,x_0)}(g(t))$$

To do so, we apply Itô's formula and Dambis (1965), Dubins-Schwarz (1965) theorem ■

This relation is widely used in finance, for instance in Geman and Yor (1993) or Delbaen and Shirakawa (2002).

Let us now introduce the commonly called CEV (Constant Elasticity of Variance), which was introduced by Cox (1975, 1996) and that solves the following equation:

$$dX_t = \mu X_t dt + \sigma X_t^{\alpha} dW_t \tag{2.3}$$

with $X_0 = x_0 > 0$, $\alpha \in \mathbb{R}$, $\mu \in \mathbb{R}$, $\sigma > 0$ and W_t a standard brownian motion.

Lemma 2.2.2 *A CEV Process X_t which solves equation (2.3) can be represented as a power of a CIR process, indeed for $\beta = 2(\alpha - 1)$, $1/X_t^{\beta}$ solves*

$$d(\frac{1}{X_t^{\beta}}) = \left(a - b\frac{1}{X_t^{\beta}}\right)dt + \Sigma\sqrt{|\frac{1}{X_t^{\beta}}|}dW_t \tag{2.4}$$

where $a = \frac{\beta(\beta+1)\sigma^2}{2}$, $b = \beta\mu$, $\Sigma = -\beta\sigma$ and .

Proof. This lemma is just an application of Itô's Lemma. ■

As a consequence of Lemma 2.2.1 and Lemma 2.2.2, one obtains the following representation for a CEV process:

Proposition 2.2.3 *A CEV Process X_t solution of equation (2.3) can be represented in the following form:*

$$X_t = e^{\mu t} BESQ^{\frac{1}{2(1-\alpha)}}_{(\frac{2\alpha-1}{\alpha-1}, x_0^{-2(\alpha-1)})}\left(\frac{(\alpha-1)\sigma^2}{2\mu}(e^{2(\alpha-1)\mu t} - 1)\right) \tag{2.5}$$

where $BESQ_{(\delta,x_0)}$ denotes a squared Bessel Process starting from x_0 at time $t = 0$ of dimension δ.

2.2.2 Distributions and Boundaries

We will now recall well known results about squared Bessel processes and deduce some properties about CEV processes.

Path Properties

Proposition 2.2.4 *According to its dimension, the squared Bessel process has different properties:*
(i) if $\delta \leq 0$, 0 is an absorbing point.
(ii) if $\delta < 2$, {0} is reached a.s.
(iii) if $\delta \geq 2$, {0} is polar.
(iv) if $\delta \leq 2$, BESQ is recurrent.
(v) if $\delta > 2$, BESQ is transient.
(vi) if $0 < \delta < 2$, {0} is instantaneously reflecting.

Proof. The proof can be found in Revuz and Yor (2001) chapter XI. ∎

As a consequence, one may give some properties of the CEV diffusions. A topic of interest for the remaining of the paper is whether or not $\{0\}$ is reached by a CEV process.

Proposition 2.2.5 *According to the value of α, the CEV diffusion has different properties:*
(i) if $\alpha < 1$, {0} is reached a.s.
(ii) if $\alpha \leq \frac{1}{2}$, {0} is instantaneously reflecting.
(iii) if $\frac{1}{2} < \alpha < 1$, {0} is an absorbing point.
(iv) if $\alpha \geq 1$, {0} is an unreachable boundary.

Proof. It is a consequence of the previous proposition and of Proposition 2.2.3. ∎

Distributional Properties

It is important to notice that the law of a squared Bessel process can be seen in terms of non-central chi-square density:

Lemma 2.2.6 *For any $BESQ_{\delta,x}$, one has:*

$$BESQ_{\delta,x}(t) \overset{(d)}{=} tV^{(\delta,\frac{x}{t})} \tag{2.6}$$

where $V^{(a,b)}$ is a non-central chi-square r.v. with a degrees of freedom and non-centrality parameter $b \geq 0$. Its density is given by:

$$f_{a,b}(v) = \frac{1}{2^{\frac{a}{2}}} \exp\Big(-\frac{1}{2}(b+v)\Big) v^{\frac{a}{2}-1} \sum_{n=0}^{\infty} \Big(\frac{b}{4}\Big)^n \frac{v^n}{n!\Gamma(\frac{a}{2}+n)} \tag{2.7}$$

Proof. This proof results from simple properties of Laplace transforms and can be found for instance in Delbaen and Shirakawa (2002). ∎

We leave to the reader the calculation of the CEV density in terms of non-central chi-square distributions.

Let us recall a useful result for the remaining of the paper on the moments of a squared Bessel process:

Corollary 2.2.7 *If $V^{(a,b)}$ is a non-central chi-square r.v. with a degrees of freedom and noncentrality parameter $b \geq 0$, then for any real constants c and d:*

$$\mathbb{E}[(V^{(a,b)})^c \mathbf{1}_{\{V^{(a,b)} \geq d\}}] = e^{-\frac{b}{2}} 2^c \sum_{n \geq 0} \big(\frac{b}{2}\big)^n \frac{\Gamma(n+\frac{a}{2}+c)}{n!\Gamma\big(\frac{a}{2}+n\big)} G(n+\frac{a}{2}+c, \frac{d}{2}) \tag{2.8}$$

where G is defined as follows:

$$G(x,y) = \int_{z \geq y} \frac{z^{x-1}e^{-z}}{\Gamma(x)} \mathbf{1}_{\{z>0\}} dz$$

Proof. This calculation is a simple application of Lemma 2.2.6. ■

Finally, for the computations involved in this paper, one recalls the two following identities on the complementary non-central chi-square distribution function Q that one can find in Johnson and Kotz (1970):

$$\begin{aligned} Q(2z, 2\nu, 2\kappa) &= \sum_{n \geq 1} g(n, \kappa) G(n + \nu - 1, z) \\ 1 - Q(2\kappa, 2\nu - 2, 2z) &= \sum_{n \geq 1} g(n + \nu - 1, \kappa) G(n, z) \end{aligned}$$

where $g(x, y) = -\frac{\partial G}{\partial y}(x, y)$.

First-Hitting Times

We now concentrate on the first hitting time of 0 by a Bessel process. For this purpose, let us consider a Bessel Process R of index $\nu > 0$ starting from 0 at time 0, then, one has:

$$L_1(R) \stackrel{(d)}{=} \frac{1}{2Z_\nu} \tag{2.9}$$

where $L_1(R) = \sup\{t > 0, R_t = 1\}$ and Z_ν is a gamma variable with index ν that has the following density:

$$\mathbb{P}(Z_\nu \in dt) = \frac{t^{\nu-1} e^{-t}}{\Gamma(\nu)} \mathbf{1}_{\{t>0\}} dt \tag{2.10}$$

This result is due to Getoor (1979). Thanks to results on time reversal (see Williams (1974), Pitman and Yor (1980) and Sharpe (1980)), we have:

$$(\hat{R}_{T_0 - u}; u < T_0(\hat{R})) \stackrel{(d)}{=} (R_u; u < L_1(R)) \tag{2.11}$$

where $\hat{R}$ is a Bessel Process, starting from 1 at time 0 of dimension $\delta = 2(1 - \nu)$ and $T_0(\hat{R}) = \inf\{t > 0, \hat{R}_t = 0\}$. As a consequence, one has:

$$T_0(\hat{R}) \stackrel{(d)}{=} \frac{1}{2Z_\nu} \tag{2.12}$$

Using the scaling property of the Squared Bessel Process, one may write:

$$T_0(BESQ_x^\delta) \stackrel{(d)}{=} \frac{x}{2Z_\nu} \tag{2.13}$$

with $\delta = 2(1 - \nu)$.

Hence, we are now able to state the proposition below:

Proposition 2.2.8 *The probability of a CEV diffusion solution of equation (2.3) to reach* 0 *at time* T *with* $\alpha < 1$ *is given by:*

$$\mathbb{P}(T_0 \leq T | X_0 = x_0) = G(\frac{1}{2(1 - \alpha)}, \zeta_T) \tag{2.14}$$

where G and ξ_T are defined as follows:

$$G(x,y) = \int_{z\geq y} \frac{z^{x-1}e^{-z}}{\Gamma(x)} \mathbf{1}_{\{z>0\}} dz \tag{2.15}$$

$$\zeta_T = \frac{\mu x_0^{2(1-\alpha)}}{(1-\alpha)\sigma^2(1-e^{2(\alpha-1)\mu T})} \tag{2.16}$$

Proof. This proof is just a consequence of Proposition 2.2.3 and equation (2.13). ∎

Remark 2.2.9 *The calculation of the probability of default was originally done by Cox (1975).*

In order to compute first-passage times of scalar Markovian diffusions below a fixed level, let us recall Itô and McKean (1974) results. If $(X_t, t \geq 0)$ is scalar Markovian time-homogeneous diffusion starting from x_0 at time 0 of infinitesimal generator $\mathcal{L}$ and that we define $\tau_H = \inf\{t \geq 0; X_t \leq H\}$ for $H < x_0$, then for any $\lambda > 0$, we have

$$\mathbb{E}[e^{-\lambda\tau_H}] = \frac{\phi_\lambda(x_0)}{\phi_\lambda(H)}$$

where ϕ_λ is solution of the ODE

$$\mathcal{L}\phi = \lambda\phi$$

with the following limit conditions:
$\lim_{x\to\infty} \phi_\lambda(x) = 0$
If 0 is a reflecting boundary then $\phi_\lambda(0+) < \infty$
If 0 is an absorbing boundary then $\phi_\lambda(0+) = \infty$

As a first example, let us now consider the first-hitting time below a fixed level $0 < y \leq x$ of a Bessel process R_t of dimension $\delta = 2(\nu+1)$ starting from x :

$$\tau_y = \inf\{t \geq 0; R_t \leq y\}$$

The law of τ_y (see Itô and McKean (1974), Kent (1978) or Pitman and Yor (1980)) is obtainable from the knowledge of its Laplace transform $\mathcal{L}$. One has for any positive λ

$$\begin{aligned}\mathcal{L}(\lambda) &= \mathbb{E}[e^{-\lambda\tau_y}] \\ &= \frac{x^{-\nu}K_\nu(x\sqrt{2\lambda})}{y^{-\nu}K_\nu(y\sqrt{2\lambda})}\end{aligned}$$

where $\nu \in \mathbb{R}\backslash\mathbb{Z}$ and K_ν is a Modified Bessel function defined as follows:

$$\begin{aligned}K_\nu(x) &= \frac{\pi}{2\sin(\nu\pi)}(I_{-\nu}(x) - I_\nu(x)) \\ I_\nu(x) &= \sum_{k=0}^{\infty} \frac{(x/2)^{\nu+2k}}{k!\Gamma(\nu+k+1)}\end{aligned}$$

As a second example that will be useful for the computation of EDS prices, let us write the infinitesimal generator of a CEV process:

$$\mathcal{L}_{CEV}\phi = \sigma^2 x^{2\alpha}\frac{d^2\phi}{dx^2} + \mu x\frac{d\phi}{dx}$$

that must solve

$$\mathcal{L}_{CEV}\phi = \lambda\phi$$

with the following conditions:
ϕ_λ is a decreasing function, $\lim_{x\to\infty}\phi_\lambda(x) = 0$
If $\alpha \le \frac{1}{2}$, then $\phi_\lambda(0+) = \infty$
If $\frac{1}{2} < \alpha < 1$, then $\phi_\lambda(0+) < \infty$

We obtain the following result whose computations of the Laplace transforms were originally performed by Davydov and Linetsky (2001):

Proposition 2.2.10 *For a CEV process solution of (2.3) with $\alpha < 1$ and $\mu \neq 0$, then*

$$\phi_\lambda(x) = x^{\alpha-\frac{1}{2}}\exp\left(-\frac{\mu x^{2(1-\alpha)}}{\sigma^2(1-\alpha)}\right)W_{k,m}\left(\frac{|\mu|x^{2(1-\alpha)}}{\sigma^2(1-\alpha)}\right) \tag{2.17}$$

where

$$k = sgn(\mu)(\frac{1}{4(1-\alpha)} - \frac{1}{2}) - \frac{\lambda}{2|\mu|(1-\alpha)} \qquad \textit{and} \qquad m = \frac{1}{4(1-\alpha)}$$

and $W_{k,m}$ is a Whittaker function

The definition of the Whittaker function can be found for instance in Abramowitz and Stegun (1972).

2.2.3 Loss of Martingality

Let us now state a result on some martingale properties of Bessel processes which play an essential role in pricing theory as is well known:

Theorem 2.2.11 *Let R_t be a Bessel process of dimension δ starting from $a \neq 0$, then:*
(i) If $\delta \le 0$, $R_t^{2-\delta}$ is a true martingale up to the first hitting time of 0.
(ii) If $0 < \delta < 2$, the process $R_t^{2-\delta} - L_t$ is a martingale where L_t is a continuous increasing process carried by the zeros of $(R_t, t \ge 0)$.
(iii) If $\delta = 2$, $log(R_t)$ is a strict local martingale.
(iv) If $\delta > 2$, $R_t^{2-\delta}$ is a strict local martingale. Moreover, the default of martingality is

$$\gamma^{(\delta)}(t) = \mathbb{E}[R_0^{2-\delta}] - \mathbb{E}[R_t^{2-\delta}] = a^{2-\delta}\mathbb{P}_a^{(4-\delta)}(T_0 \le t) \tag{2.18}$$

where $\mathbb{P}_a^\delta$ is the law of $(R_t^{(\delta)}, t \ge 0)$.

Proof. (i) and (ii): Since $\{0\}$ is reached a.s., we need to apply Itô's formula in a positive neighborhood of 0. Let us consider $\epsilon > 0$. We have:

$$(\epsilon + R_t^2)^{1-\frac{\delta}{2}} = (\epsilon + a^2)^{1-\frac{\delta}{2}} + (2-\delta)\int_0^t (\epsilon + R_s^2)^{-\frac{\delta}{2}} R_s dW_s + \epsilon\delta(1-\frac{\delta}{2})\int_0^t \frac{ds}{(\epsilon + R_s^2)^{\frac{\delta}{2}+1}}$$

Then, as ϵ tends to zero, it is easy to see the first term of the right hand side is a true martingale for $\delta < 2$ and that the second term of the right hand side is increasing whose support is the zeros of $(R_t, t \geq 0)$ when $\delta \geq 0$. If $\delta < 0$, $T_t^{2-\delta}$ is a true martingale.
(iii): By applying Itô formula, we obtain

$$\log(R_t) = \log(R_0) + \int_0^t \frac{dW_s}{R_s}$$

We then see that $\log(R_t)$ is a local martingale. We prove that it is a strict local martingale by first using the fact that

$$\mathbb{P}_a^\delta = \mathbb{P}_0^\delta * \mathbb{P}_a^0$$

then writing that

$$\begin{aligned}\mathbb{E}[\log(R_t)] &= \frac{1}{2}\mathbb{E}[\log(R_t^2)] = \frac{1}{2}\mathbb{E}[\log(BESQ_{2,0}(t) + BESQ_{0,a}(t))] \\ &\geq \frac{1}{2}\mathbb{E}[\log(BESQ_{2,0}(t))]\end{aligned}$$

and finally since $BESQ_{2,0}(t) \overset{d}{=} 2t\mathbf{e}$ where $\mathbf{e}$ is a standard exponential, we obtain

$$\mathbb{E}[\log(R_t)] \geq C + \frac{1}{2}\log(t) \longrightarrow_{t\to\infty} +\infty$$

which shows that $\log(R_t)$ is not a true martingale.
(iv): To compute $\gamma^{(\delta)}$, we will need the following result:

Lemma 2.2.12 *Let $(R_t^{(\delta)}, t \geq 0)$ be a Bessel process of dimension $\delta > 2$ starting from $a \neq 0$, then*

$$\mathbb{P}^{4-\delta}_{a|\mathcal{R}_t \cap \{t<T_0\}} = \left(\frac{R_t^{(\delta)}}{a}\right)^{2-\delta} \cdot \mathbb{P}^\delta_{a|\mathcal{R}_t}$$

where $\mathcal{R}_t$ is the canonical filtration of the Bessel process and T_0 the first-hitting time of the level 0.

Proof. This property results from a double application of Girsanov Theorem by computing

$$\frac{d\mathbb{P}^{4-\delta}_{a|\mathcal{R}_t\cap\{t<T_0\}}}{d\mathbb{P}^2_{a|\mathcal{R}_t}} \quad \text{and} \quad \frac{d\mathbb{P}^\delta_{a|\mathcal{R}_t}}{d\mathbb{P}^2_{a|\mathcal{R}_t}}$$

Then, by identification, one gets the announced result. A more general result can be found in Yor (1992). ■

We may then write

$$\mathbb{E}^{(\delta)}[R_t^{2-\delta}] = \mathbb{E}^{(4-\delta)}[a^{2-\delta}\mathbf{1}_{\{t<T_0\}}]$$

and consequently compute the default of martingality. ∎

A proof in the case $0 < \delta < 2$ can be found in Donati-Martin et al. (2006) and proofs when $\delta > 2$ exist in Elworthy, Li and Yor (1999). As a consequence, we obtain similar results for a CEV process.

Proposition 2.2.13 *Let X_t be a CEV Process of elasticity α solving the following equation*

$$dX_t = \mu dt + \sigma X_t^\alpha dW_t$$

then:
(i) If $\alpha \leq \frac{1}{2}$, the process $e^{-\mu t}X_t$ is a true martingale up to the first hitting time of 0.
(ii) If $\frac{1}{2} < \alpha < 1$, the process $e^{-\mu t}X_t - L_t^X$ is a martingale where L_t^X is a continuous increasing process carried by the zeros of $(X_t, t \geq 0)$ and consequently $e^{-\mu t}X_t$ is a true martingale up to the first hitting time of 0.
(ii) If $\alpha = 1$, $e^{-\mu t}X_t$ is a geometric Brownian motion and hence a martingale.
(iii) If $\alpha > 1$, $e^{-\mu t}X_t$ is a strictly local martingale. Moreover, the default of martingality is

$$\gamma_X(t) = \mathbb{E}[X_0] - \mathbb{E}[e^{-\mu t}X_t] = x_0 G(\frac{1}{2(\alpha-1)}, \zeta_T) \tag{2.19}$$

where G and ζ_T are defined as follows:

$$\begin{aligned} G(x,y) &= \int_{z \geq y} \frac{z^{x-1}e^{-z}}{\Gamma(x)} \mathbf{1}_{\{z>0\}} dz \\ \zeta_T &= \frac{\mu x_0^{2(1-\alpha)}}{(\alpha-1)\sigma^2(e^{2(\alpha-1)\mu T} - 1)} \end{aligned}$$

Proof. This is just an application of Theorem 2.2.11, Proposition 2.2.3 and equation (2.14). ∎

A proof of the failure of the martingale property can be found in Lewis (2000).

Remark 2.2.14 *For $\alpha > 1$, one has $\forall (t, K) \in \mathbb{R}_+^2$:*

$$\mathbb{E}[(e^{-\mu t}X_t - K)_+] - \mathbb{E}[(K - e^{-\mu t}X_t)_+] + \gamma_X(t) = \mathbb{E}[X_0] - K \tag{2.20}$$

The last equation shows that in the case of a strictly local martingale, the Call price must incorporate the default of martingality in order to remain in a No Arbitrage model. For a study on option pricing for strict local martingales, we refer to Madan and Yor (2006) for continuous processes and to Chybiryakov (2006) for jump-diffusion processes. Lewis (2000) also did this study in the case of explosions with stochastic volatility models and in particular for a CEV diffusion.

2.3 Credit-Equity Modelling

2.3.1 Model Implementation

Usually, in the mathematical finance literature, one defines a CEV diffusion for the stock price dynamics S to be

$$\frac{dS_t}{S_t} = \mu dt + \sigma S_t^{\alpha-1} dW_t$$

First of all, in a credit perspective, we will just consider the case $\alpha < 1$ since we are interested in models with a non-zero probability of default. Once the stock has reached zero, the firm has bankrupted and that is the reason why we stop the CEV diffusion at its first default time. Then from what has been proven above, we know that the stock price process hence defined is a true martingale and that ensures the Absence of Arbitrage and moreover the uniqueness of the solution. Hence, the stock price diffusion now becomes under the risk-neutral pricing measure:

$$\frac{dS_t}{S_t} = rdt + \sigma S_t^{\alpha-1} dW_t \quad \text{if} \quad t < \tau.$$

$$S_t = 0 \quad \text{if} \quad t \geq \tau.$$

where $\tau = T_0(S) = \inf\{t > 0, S_t = 0\}$. In other words, the stock price process considered is nothing else than a stopped CEV diffusion $(S_{t\wedge\tau})_{t\geq 0}$.

Remark 2.3.1 *Delbaen and Shirakawa (2002) showed the existence of a risk-neutral probability measure whose uniqueness is only ensured on the stock price filtration considered at time τ $\mathcal{F}_\tau = \sigma(S_t, t \leq \tau)$. Since our purpose is to compute the price of options whose payoffs are $\mathcal{F}_\tau$−measurable, we have the uniqueness of the no-arbitrage probability.*

2.3.2 European Vanilla Option Pricing

Lemma 2.2.12 states that

$$\mathbb{P}^{4-\delta}_{x|\mathcal{R}_t \cap \{t<T_0\}} = \left(\frac{R_t^{(\delta)}}{x}\right)^{2-\delta} \cdot \mathbb{P}^{\delta}_{x|\mathcal{R}_t} \tag{2.21}$$

Thanks to this identity, we obtain the law of the stopped CEV diffusion at a given time. Lemma 2.2.6 and Corollary 2.2.7 enable us to compute the call and put option price:

For the call C_0 option price

$$\begin{aligned} C_0 &= e^{-rT}\mathbb{E}[(S_{T\wedge\tau} - K)_+] \\ &= e^{-rT}\mathbb{E}[(S_T - K)_+ \mathbf{1}_{T<\tau}] \end{aligned}$$

and the put P_0 option price:

$$\begin{aligned} P_0 &= e^{-rT}\mathbb{E}[(K - S_{T\wedge\tau})_+] \\ &= e^{-rT}\mathbb{E}[(K - S_T)_+ \mathbf{1}_{T<\tau}] + Ke^{-rT}\mathbb{P}(\tau \leq T) \end{aligned}$$

Consequently, for the call price:

$$C_0 = S_0 Q(z_T, 2 + \frac{1}{1-\alpha}, 2\zeta_T) - Ke^{-rT}(1 - Q(2\zeta_T, \frac{1}{1-\alpha}, z_T))$$

and for the put price:

$$\begin{aligned} P_0 &= Ke^{-rT}\big(Q(2\zeta_T, \frac{1}{1-\alpha}, z_T) - G(\frac{1}{2(1-\alpha)}, \zeta_T)\big) \\ &\quad - S_0(1 - Q(z_T, 2 + \frac{1}{1-\alpha}, 2\zeta_T)) + Ke^{-rT}\mathbb{P}(\tau \leq T) \\ &= Ke^{-rT} Q(2\zeta_T, \frac{1}{1-\alpha}, z_T) - S_0(1 - Q(z_T, 2 + \frac{1}{1-\alpha}, 2\zeta_T)) \end{aligned}$$

where

$$
\begin{aligned}
z_T &= \frac{2rK^{2(1-\alpha)}}{\sigma^2(1-\alpha)(e^{2(1-\alpha)rT}-1)} \\
\zeta_T &= \frac{rS_0^{2(1-\alpha)}}{(1-\alpha)\sigma^2(1-e^{-2(1-\alpha)rT})}
\end{aligned}
$$

Hence, one easily verifies that the put-call parity is satisfied. Closed-form CEV option pricing formulas were originally computed by Cox (1975) for $\alpha < 1$ and Schroder (1989) expressed those formulas in terms of non-central chi-square distributions. Computing option prices using the squared Bessel processes distributions was done by Delbaen and Shirakawa (2002).

2.3.3 Pricing of Credit and Equity Default Swaps

Since we are dealing with default probabilities, it is obvious to consider derivative products relying on these probabilities. One of the most liquid protection instruments against default is the Credit Default Swap (CDS). The buyer of the protection agrees to pay periodical amounts until a default time (if it occurs) and in exchange receives a cash amount which is a notional amount minus a recovery rate in the case the company on which the contract is written, defaults. The payoff of such kind of contract is:

$$\Pi_{CDS} = -\sum_{i=1}^{n} e^{-rT_i} C \mathbf{1}_{\{\tau > T_i\}} + e^{-r\tau}(1-R)\mathbf{1}_{\{\tau \leq T_n\}}$$

where C is the periodical coupon, $T_1, ..., T_n$ the payment dates, R the recovery rate assumed to be deterministic and τ the default time. For simplicity purposes, we consider in this paper deterministic interest rates. The CDS Fair Price is the expectation of the payoff conditionally to the spot price filtration taken at the pricing time, e.g.:

$$CDS_t(T_1, T_n; C; R) = -C\sum_{i=1}^{n} e^{-r(T_i-t)}\mathbb{P}(\tau > T_i|S_t) + (1-R)\mathbb{E}[e^{-r(\tau-t)}\mathbf{1}_{\{\tau \leq T_n\}}|S_t]$$

By absence of arbitrage, one must have $CDS_t(T_1, T_n; C; R) = 0$ and then

$$C = \frac{(1-R)\mathbb{E}[e^{-r(\tau-t)}\mathbf{1}_{\{\tau \leq T_n\}}|S_t]}{\sum_{i=1}^{n} e^{-r(T_i-t)}\mathbb{P}(\tau > T_i|S_t)}$$

From Proposition 2.2.8, we know the value of $(\mathbb{P}(\tau > T_i|S_t)_{1\leq i \leq n})$. It then remains to compute the following quantity $\mathbb{E}[e^{-r\tau}\mathbf{1}_{\tau \leq t}]$ to be able to price the CDS coupon C. By an integration by parts, we show that

$$\mathbb{E}[e^{-r\tau}\mathbf{1}_{\tau \leq t}] = e^{-rt}\mathbb{P}(\tau \leq t) + r\int_0^t e^{-rs}\mathbb{P}(\tau \leq s)ds \tag{2.22}$$

Otherwise, one could just obtain this expectation by directly using the density of the first-hitting time of 0 that is provided by the differentiation of the cumulative distribution function :

$$f_\tau(t) = \frac{2r(1-\alpha)\zeta_t^{\frac{1}{2(1-\alpha)}} e^{-\zeta_t}}{\Gamma(\frac{1}{2(1-\alpha)})(e^{2(1-\alpha)rt}-1)}$$

where ζ_t is defined above.
EDSs are very similar to CDSs except that payouts occur when the stock price falls under a pre-defined level, which is often referred to as a trigger price. The trigger price is generally between 30% and 50% of the equity stock price at the beginning of the contract. Hence, these contracts provide a protection against a credit event happening on the equity market for the buyer. They were initiated by the end of 2003. At that time, it had become difficult in many countries to structure investment-grade credit portfolios with good returns because the CDS spreads were tightening, as reported by Sawyer (2003). Another reason why people have interest in those contracts is because the settlement of the default is directly observed on the stock price. Let us now define τ_L as the first passage time of the stock price process under the level $L < S_0$. Formally, we write $\tau_L = \inf\{t > 0; S_t \leq L\}$. We recall the general valuation formula of an EDS:

$$EDS_t(T_1, T_n; C; R) = -C \sum_{i=1}^{n} e^{-r(T_i - t)} \mathbb{P}(\tau_L > T_i | S_t) + \mathbb{E}[e^{-r(\tau_L - t)} \mathbf{1}_{\{\tau_L \leq T_n\}} | S_t]$$

where C is the coupon, $T_1, ..., T_n$ the payment dates and r the risk-free interest rate. Again, by absence of arbitrage, we can find the coupon price, by stating that at the initiation of the contract:

$$EDS_{t=0}(T_1, T_n; C; R) = 0$$

Or equivalently

$$C = \frac{\mathbb{E}[e^{-r(\tau_L - t)} \mathbf{1}_{\{\tau_L \leq T_n\}} | S_t]}{\sum_{i=1}^{n} e^{-r(T_i - t)} \mathbb{P}(\tau_L > T_i | S_t)}$$

In order to price the coupon C, one needs to evaluate:

$$\mathbb{E}[e^{-r\tau_L} \mathbf{1}_{\{\tau_L \leq t\}}] \quad \text{and} \quad \mathbb{P}(\tau_L \leq t)$$

An integration by parts gives the Laplace transform of $\mathbb{P}(\tau_L \leq t)$ for any $\lambda > 0$

$$\int_0^{+\infty} dt e^{-\lambda t} \mathbb{P}(\tau_L \leq t) = \frac{\mathbb{E}[e^{-\lambda \tau_L}]}{\lambda}$$

Applying Fubini theorem, one observes that

$$\int_0^{+\infty} dt e^{-\lambda t} \mathbb{E}[e^{-r\tau_L} \mathbf{1}_{\{\tau_L \leq t\}}] = \frac{\mathbb{E}[e^{-(r+\lambda)\tau_L}]}{\lambda}$$

Hence using Proposition 2.2.10, one is able to compute the Laplace transform of the desired quantities necessary to evaluate an EDS. One can then use numerical techniques (see Abate and Whitt (1995) for instance) to inverse the Laplace transform in order to evaluate prices.

2.4 Stochastic Volatility for CEV Processes

2.4.1 A Zero Correlation Pricing Framework

Impact of a Stochastic Time Change

Due to the very important dependency between the probability of default, the level of volatility and the skewness, we were naturally brought to consider extensions of the CEV model that could relax the high correlation between these three effects. More precisely, in a CEV model, if one first calibrates the implied at-the-money volatility, then either the skewness or the CDS will be calibrated on adjusting the elasticity parameter. Hence, to be able to get some freedom on the volatility surface, a possible extension is to introduce a stochastic volatility in the CEV model instead of a constant volatility. A CEV diffusion with a stochastic volatility is actually just a power of a squared Bessel Process with a stochastic time change instead of having a deterministic one like in Proposition 2.2.3.

Another extension is to consider a power of a Bessel Process time changed by an independent increasing process. More precisely, one writes the following process for the stock price:

$$S_t = e^{rt} BESQ_{(\delta,x)}^{1-\frac{\delta}{2}}(\xi_t) \quad if \quad t < \tau. \tag{2.23}$$

$$S_t = 0 \quad if \quad t \geq \tau.$$

where $x = S_0^{\frac{2}{2-\delta}}$, $\tau = T_0(S) = \inf\{t > 0, S_t = 0\} = \xi^{-1}(T_0(BESQ))$ and ξ_t is a strictly increasing continuous integrable process independent from the squared Bessel process. Subordinating a continuous process by an independent Lévy process is an idea that goes back to Clark (1973). Stochastic time changes are somehow equivalent to adding a stochastic volatility in stock price diffusions. The basic intuition underlying this approach could be foreseen through the scaling property of the Brownian motion, or through Dambis (1965),Dubins and Schwarz (1965) (DDS) theorem or even its extension to semimartingales by Monroe (1978). More recently, Carr et al. (2003) generated uncertainty by speeding up or slowing down the rate at which time passes with a Lévy process. Our approach differs from the one done in the Lévy processes literature for mathematical finance: We are not considering the exponential of a time changed Lévy process but a power of a time changed Bessel process. Thanks to Lamperti representation (1972), this means that we are considering a time changed geometric Brownian motion B. More precisely, it is known that

$$R_t = \exp\big(B_{C_t} + \nu C_t\big) \quad \text{and} \quad C_t = \int_0^t \frac{ds}{R_s^2}$$

where $(R_t, t \geq 0)$ is a Bessel process of dimension $\delta = 2(1+\nu)$ starting from $a \neq 0$. Hence the time change considered in the stock price is

$$Y_t = \int_0^{\xi_t} \frac{ds}{R_s^2}$$

and the stock price process as defined in equation (2.23) can be identified as follows:

$$S_t = e^{rt} \exp\Big(- 2\nu B_{Y_t} - \frac{(2\nu)^2}{2} Y_t\Big)$$

As a consequence, we have now proposed a new class of time changes where analytical computations are possible thanks to a good knowledge of Bessel processes.

For the absence of arbitrage property, there must exist a probability under which all the actualized stock prices are martingales. A very simple property on martingales is that a process M_t is a martingale if and only if for every bounded stopping time T, $\mathbb{E}[M_T] = \mathbb{E}[M_0]$. Nonetheless, this result is not very convenient. Let us state and give a straightforward proof of the martingality of the stock price process

Proposition 2.4.1 *Consider $M_t = BESQ^{1-\frac{\delta}{2}}_{(\delta,x)}(\xi_{t\wedge\tau})$ where following the previous hypotheses ξ_t is a strictly increasing continuous integrable process independent from $BESQ$, τ is the $(M_t, t \geq 0)$ first hitting time of 0 and $BESQ_{(\delta,x)}$ is a squared Bessel process of dimension $\delta < 2$ starting from $x \neq 0$, then $(M_t, t \geq 0)$ is a true martingale.*

Proof. Let us define $\mathcal{R}_t = \sigma(R_s; s \leq t)$. We then naturally write the canonical filtrations $\mathcal{R}_{\xi_t} = \sigma(R_{\xi_s}; s \leq t)$ and $\Xi_t = \sigma(\xi_s; s \leq t)$. For any bounded functional F, we want to compute

$$\mathbb{E}[F\big(R_{\xi_u}; u \leq s\big)\big(R^{2-\delta}_{\xi_t} - R^{2-\delta}_{\xi_s}\big)]$$

Since ξ is integrable and independent from R, we obtain by using Fubini theorem

$$\begin{aligned}
\mathbb{E}[F\big(R_{\xi_u}; u \leq s\big)\big(R^{2-\delta}_{\xi_t} - R^{2-\delta}_{\xi_s}\big)] &= \mathbb{E}\Big[\mathbb{E}[F\big(R_{\xi_u}; u \leq s\big)\big(R^{2-\delta}_{\xi_t} - R^{2-\delta}_{\xi_s}\big) \big| \, \Xi_t]\Big] \\
&= \int \mathbb{P}_{\Xi_t}(da)\mathbb{E}[F\big(R_{a(u)}; u \leq s\big)\big(R^{2-\delta}_{a(t)} - R^{2-\delta}_{a(s)}\big)]
\end{aligned}$$

The latest quantity is null by Theorem 2.2.11 and we have then shown that for $s \leq t < \tau$

$$R^{2-\delta}_{\xi_s} = \mathbb{E}[R^{2-\delta}_{\xi_t} | \mathcal{R}_{\xi_s}]$$

which is the announced result. ■

Pricing Vanilla Options

One can find closed-form formulas for the call and put options prices. Let us define the two following quantities $C_0(x, \delta, K, T; S_0)$ and $P_0(x, \delta, K, T; S_0)$:

$$\begin{aligned}
C_0(x, \delta, K, T; S_0) &= S_0 Q\Big(\frac{(Ke^{-rT})^{\frac{2}{2-\delta}}}{x}, 4-\delta, \frac{S_0^{\frac{2}{2-\delta}}}{x}\Big) \\
&\quad - Ke^{-rT}\Big(1 - Q\Big(\frac{S_0^{\frac{2}{2-\delta}}}{x}, 2-\delta, \frac{(Ke^{-rT})^{\frac{2}{2-\delta}}}{x}\Big)\Big) \\
P_0(x, \delta, K, T; S_0) &= Ke^{-rT} Q\Big(\frac{S_0^{\frac{2}{2-\delta}}}{x}, 2-\delta, \frac{(Ke^{-rT})^{\frac{2}{2-\delta}}}{x}\Big) \\
&\quad - S_0\Big(1 - Q\Big(\frac{(Ke^{-rT})^{\frac{2}{2-\delta}}}{x}, 4-\delta, \frac{S_0^{\frac{2}{2-\delta}}}{x}\Big)\Big)
\end{aligned}$$

From there, one may obtain the option prices under the new general framework.

Proposition 2.4.2 *If one has the following stock price process:*

$$S_t = e^{rt} BESQ^{1-\frac{\delta}{2}}_{(\delta,x)}(\xi_t) \quad if \quad t < \tau.$$

$$S_t = 0 \quad if \quad t \geq \tau.$$

where $x = S_0^{\frac{2}{2-\delta}}$, $\tau = T_0(S) = \inf\{t > 0, S_t = 0\}$ *and* ξ_t *is a strictly increasing continuous integrable process independent from* $BESQ$ *whose probability measure is* $\mu_{\xi_t}(dx)$*, then:*

$$\begin{aligned} C_0 &= \int_{\mathbb{R}_+} C_0(x, \delta, K, T; S_0)\mu_{\xi_T}(dx) \\ P_0 &= \int_{\mathbb{R}_+} P_0(x, \delta, K, T; S_0)\mu_{\xi_T}(dx) \end{aligned}$$

Proof. Let us prove this result for the call option price, a similar result may be obtained for the put price. One has:

$$\begin{aligned} C_0 &= e^{-rT}\mathbb{E}[(S_T - K)_+] \\ &= e^{-rT}\mathbb{E}\big(\mathbb{E}[(S_T - K)_+|\sigma(\xi_s; s \le T)]\big) \\ &= \mathbb{E}[C_0(\xi_T, \delta, K, T; S_0)] \end{aligned}$$

■

Computing the Default

Having the integrability of the change of time and knowing its density, one could find a closed-form formula for the probability of default $\tau = T_0(S) = \inf\{t > 0, S_t = 0\}$ where $S_t = e^{rt}BESQ_{(\delta,x)}^{1-\frac{\delta}{2}}(\xi_t)$. Let us now compute the probability of default the proof of which is left to the reader:

Proposition 2.4.3 *If one considers a stock price process defined as follows:*

$$S_t = e^{rt}BESQ_{(\delta,x)}^{1-\frac{\delta}{2}}(\xi_t)$$

then the probability of default $\tau = \inf\{t > 0, S_t = 0\}$ *is given by*

$$\mathbb{P}(\tau \le T) = \mathbb{E}\left[G(1 - \frac{\delta}{2}, \frac{S_0^{\frac{2}{2-\delta}}}{\xi_T})\right]$$

where G *is the complementary Gamma function.*

2.4.2 CESV Models

Stochastic volatility models were used in a Black and Scholes (1973) and Merton (1973) framework mainly to capture skewness and kurtosis effects, or in terms of implied volatility skew and smile. In a Constant Elasticity of Variance framework, one would use stochastic volatility not to capture the leverage effect which partly already exists due to the elasticity parameter but to obtain environments for instance of low volatilities, high probabilities of default and low skew. Let us consider a jump-diffusion process $(\sigma_t, t \ge 0)$ satisfying $\sup_{0\le t\le T} E[\sigma_t^2] < \infty$ for all stopping time T to model the volatility. We will call those diffusions Constant Elasticity of Stochastic Variance (CESV) for the remainder of the paper. Leblanc (1997) introduced stochastic volatility for CEV processes.

Hence, the class of models under a risk-neutral probability measure proposed is of the following form:

$$\frac{dS_t}{S_t} = rdt + \sigma_{t-} S_t^{\alpha-1} dW_t$$

where σ is assumed to be independent from the Brownian motion driving the stock price returns. Next, within an equity subject to bankruptcy framework, we are going to stop the diffusion when the stock reaches 0 just as in the previous section. As a consequence, our diffusion becomes:

$$\begin{aligned} \frac{dS_t}{S_t} &= rdt + \sigma_{t-} S_t^{\alpha-1} dW_t \quad if \quad t < \tau. \\ S_t &= 0 \quad if \quad t \geq \tau. \end{aligned}$$

where $\tau = T_0(S) = \inf\{t > 0, S_t = 0\}$.
Before giving any concrete examples, let us show how CESV models can be seen as Bessel processes with a stochastic time change. So first, let us recall elementary results:

Lemma 2.4.4 *Let R be a time change with $s \mapsto R_s$ continuous, strictly increasing, $R_0 = 0$ and $R_t < \infty$, for each $t \geq 0$, then for any continuous semimartingale X and any caglad (left continuous with right limits) bounded adapted process H, one has:*

$$\int_0^{R_t} H_s dX_s = \int_0^t H_{R_u} dX_{R_u} \tag{2.24}$$

Proof. The proof can be found in Revuz and Yor (2001). ∎

Then, using Lemma 2.4.4, (DDS) theorem and Itô formula, we obtain that

$$\begin{aligned} S_t &\overset{d}{=} e^{rt} BESQ^{\frac{1}{2(1-\alpha)}}_{(2-1/(1-\alpha), S_0^{2(1-\alpha)})}(H_{t\wedge\tau}) \\ \tau &= \inf\{t \geq 0, S_t = 0\} \\ H_t &= (1-\alpha)^2 \int_0^t \sigma_s^2 e^{-2(1-\alpha)rs} ds \end{aligned}$$

H_t is by construction an increasing continuous integrable process.
Hence $(e^{-rt} S_t, t \geq 0)$ is a continuous martingale by Proposition 2.4.1. All the results of the previous subsection apply and we are able to compute Vanilla option and CDS prices conditionally on the knowledge of the law of H_t. As a result, we showed that a CESV model is in fact a timed-changed power of Bessel process where the subordinator is an integrated time change $H_t = \int_0^t h_s ds$ with a specific rate of time change h_t that is defined by

$$h_t = (1-\alpha)^2 \sigma_t^2 e^{-2(1-\alpha)rt}$$

We now provide two examples of well-known stochastic volatility models where we compute the law of the time change.

Heston Model Let us first consider a CIR (1985) diffusion for the volatility process

$$d\sigma_t^2 = \kappa(\theta - \sigma_t^2)dt + \eta\sigma_t dW_t^\sigma \qquad \text{and} \qquad \sigma_0^2 = x > 0$$

where κ, θ and η are strictly positive constants and W^σ is a Brownian motion independent from W. In fact we are proposing a variation of the Heston (1993) model by considering $\alpha \neq 1$. We then want to compute the law of

$$H_t = (1-\alpha)^2 \int_0^t \sigma_s^2 e^{-2(1-\alpha)rs} ds$$

More precisely, we will compute its Laplace transform, that is to say, for any $\lambda > 0$

$$\mathbb{E}[e^{-\lambda H_t}]$$

For this purpose, let us use the following result:

Lemma 2.4.5 *If X a squared Bessel process $BESQ_{(\delta,x)}$ starting from $x \neq 0$ and of dimension δ, then for any function $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that for any $t > 0$: $\int_0^t f(s)ds < \infty$, we have*

$$\mathbb{E}\Big[\exp\big(-\int_0^t X_s f(s)ds\big)\Big] = \frac{1}{\psi_f'(t)^{\delta/2}} \exp \frac{x}{2}\Big(\phi_f'(0) - \frac{\phi_f'(t)}{\psi_f'(t)}\Big)$$

where ϕ_f is the unique solution of the Sturm-Liouville equation

$$\phi_f''(s) = 2f(s)\phi_f(s)$$

where $s \in [0;\infty[$, $\phi_f(0) = 1$, ϕ_f is positive and non-increasing and

$$\psi_f(t) = \phi_f(t) \int_0^t \frac{ds}{\phi_f^2(s)}$$

Proof. The proof can be found in Pitman and Yor (1982). ∎

By Lemma 2.2.1, we can see that

$$H_t \overset{d}{=} \big(\frac{2(1-\alpha)}{\eta}\big)^2 \int_0^{\frac{\eta^2}{4\kappa}(e^{\kappa t}-1)} X_u \frac{du}{\big(\frac{4\kappa u}{\eta^2}+1\big)^{2[\frac{(1-\alpha)r}{\kappa}+1]}}$$

where X is a $BESQ_{(\frac{4\kappa\theta}{\eta^2},x)}$. Hence for any $\lambda > 0$,

$$\mathbb{E}[e^{-\lambda H_t}] = \mathbb{E}\Big[\exp\big(-\int_0^{l(t)} X_s f_\lambda(s)ds\big)\Big]$$

with

$$l(t) = \frac{\eta^2}{4\kappa}(e^{\kappa t}-1) \qquad \text{and} \qquad f_\lambda(t) = \lambda\big(\frac{2(1-\alpha)}{\eta}\big)^2\big(\frac{4\kappa u}{\eta^2}+1\big)^{-2[\frac{(1-\alpha)r}{\kappa}+1]} \tag{2.25}$$

Defining $a = 8((1-\alpha)/\eta)^2$, $b = 4\kappa/\eta^2$ and $n = -2(\frac{(1-\alpha)r}{\kappa} + 1)$ and using Lemma 2.4.5 we are brought to the resolution of the following ordinary differential equation

$$\phi''(x) - a\lambda(bx+1)^n\phi(x) = 0$$

Then under the boundary conditions, one obtains (see Polyanin and Zaitsev (2003)):

$$\begin{aligned} \phi_\lambda(x) &= \sqrt{bx+1}\,\frac{\frac{\pi}{\sin(\nu\pi)} I_{1/(n+2)}\big(\frac{2\sqrt{a\lambda}}{b(n+2)}(bx+1)^{(n+2)/2}\big)}{I_{-1/(n+2)}\big(\frac{2\sqrt{a\lambda}}{b(n+2)}\big)} \\ &\quad + \sqrt{bx+1}\frac{K_{1/(n+2)}\big(\frac{2\sqrt{a\lambda}}{b(n+2)}(bx+1)^{(n+2)/2}\big)}{I_{-1/(n+2)}\big(\frac{2\sqrt{a\lambda}}{b(n+2)}\big)} \end{aligned} \tag{2.26}$$

$$\begin{aligned} \psi_\lambda(x) &= C_1\sqrt{bx+1}\; I_{1/(n+2)}\big(\frac{2\sqrt{a\lambda}}{b(n+2)}(bx+1)^{(n+2)/2}\big) \\ &\quad + C_2\sqrt{bx+1}\; K_{1/(n+2)}\big(\frac{2\sqrt{a\lambda}}{b(n+2)}(bx+1)^{(n+2)/2}\big) \end{aligned} \tag{2.27}$$

where with using the fact that $I'_\nu(x)K_\nu(x) - I_\nu(x)K'_\nu(x) = 1/x$ one has

$$\begin{aligned} C_1 &= -\frac{b(n+2)}{2a\lambda} K_{1/(n+2)}\big(\frac{2\sqrt{a\lambda}}{b(n+2)}\big) \\ C_2 &= \frac{b(n+2)}{2a\lambda} I_{1/(n+2)}\big(\frac{2\sqrt{a\lambda}}{b(n+2)}\big) \end{aligned}$$

We finally obtain the Laplace transform of H_t for any $\lambda > 0$

$$\mathbb{E}[e^{-\lambda H_t}] = \frac{1}{\psi'_\lambda(l(t))^{\delta/2}} \exp\frac{x}{2}\left(\phi'_\lambda(0) - \frac{\phi'_\lambda(l(t))}{\psi'_\lambda(l(t))}\right)$$

with $\delta = \frac{4\kappa\theta}{\eta^2}$.
A simpler example for the forward contract is provided in Atlan and Leblanc (2005).

Hull and White Model Let us now consider the Hull and White (1987) volatility diffusion that is driven by the following stochastic differential equation:

$$\frac{d\sigma_t^2}{\sigma_t^2} = \theta dt + \eta dW_t^\sigma$$

where θ and η are positive constants and W^σ is a Brownian motion independent from W. Then H may be computed and after some simplifications, we obtain:

$$H_t = \frac{4(1-\alpha)\sigma_0^2}{\eta^2}\int_0^{\frac{\eta^2 t}{4}} ds e^{2(W_s^\sigma + \nu s)} \tag{2.28}$$

where $\nu = \frac{2}{\eta^2}(\theta - \frac{\eta^2}{2} - 2(1-\alpha)r)$.
If we define $A_t^\nu = \int_0^t \exp 2(B_s + \nu s) ds$ where B is a Brownian motion, we recognize a typical quantity used for the pricing of Asian options with analytical formulae. Thus, we can write

$$H_t = \frac{4(1-\alpha)\sigma_0^2}{\eta^2} A^\nu_{\frac{\eta^2 t}{4}}$$

and obtain its law using Yor (1992), more precisely we have $\forall (u,v) \in \mathbb{R}_+^2$:

$$f_{|A_t^\nu}(u) = \frac{\exp\left(\frac{\pi^2}{2t} - \frac{\nu^2 t}{t} - \frac{1}{2u}\right)}{u^2\sqrt{2\pi^3 t}} \int_{-\infty}^{+\infty} dx e^{x(\nu+1)} e^{-\frac{e^{2x}}{2u}} \psi_{\frac{e^x}{u}}(t) \tag{2.29}$$

where:

$$\psi_r(v) = \int_0^\infty dy \exp(-\frac{y^2}{2v}) e^{-r\cosh(y)} \sinh(y) \sin(\frac{\pi y}{v}) \tag{2.30}$$

2.4.3 Subordinated Bessel Models

Another way to build stochastic volatility models is to make time stochastic. Geman, Madan and Yor (2001) recognize that asset prices may be viewed as Brownian motions subordinated by a random clock. The random clock may be regarded as a cumulative measure of the economic activity as said in Clark (1973) and as estimated in Ané and Geman (2000). The time must be an increasing process, thus it could either be a Lévy subordinator or a time integral of a positive process. In this paper, we only consider the case of a time integral because we need the continuity of the time change in order to compute the first-passage time at 0 to be able to provide analytical formulas for CDS prices. More generally, for the purpose of pricing path-dependent options, one needs the continuity of the time change in order to simulate increments of the time changed Bessel process. Consequently, we study the case of a time change Y_t such as

$$Y_t = \int_0^t y_s ds$$

where the rate of time change $(y_t, t \geq 0)$ is a positive stochastic process.
As we have seen in the previous subsection, considering a stochastic volatility $(\sigma_t, t \geq 0)$ in the CEV diffusion is equivalent to the following rate of time change

$$y_t = \frac{\sigma_t^2 e^{\frac{2rt}{\delta-2}}}{(2-\delta)^2}$$

where δ is the dimension of the squared Bessel process. Hence, in order to provide frameworks where one is able to compute the law of the time change, we are going to go directly through different modellings of the rate of time change y_t.

Integrated CIR Time change As a first example, let us consider the case where y_t solves the following diffusion

$$dy_t = \kappa(\theta - y_t)dt + \eta\sqrt{y_t} dW_t^Y$$

where W^Y is independent from the driving Bessel process. The Laplace transform of Y_t is then defined for any $\lambda > 0$ by :

$$\begin{aligned}\mathbb{E}[e^{-\lambda Y_t}] &= e^{\frac{\kappa^2\theta t}{\eta^2}} \frac{\exp\big(-2\lambda y_0/(\kappa+\gamma\coth(\gamma t/2))\big)}{\big(\cosh(\gamma t/2)+\frac{\kappa}{\gamma}\sinh(\gamma t/2)\big)^{2\kappa\theta/\eta^2}} \\ \gamma &= \sqrt{\kappa^2+2\eta^2\lambda}\end{aligned}$$

Integrated Ornstein-Uhlenbeck Time Change We now assume the rate of time change to be the solution of the following SDE

$$dy_t = -\lambda y_t dt + dz_t$$

where $(z_t; t \geq 0)$ is a Lévy subordinator. Let ψ_z denote the log characteristic function of the subordinator z_t, then

$$\mathbb{E}[e^{iaY_t}] = \exp\big(iay_0\frac{1-e^{-\lambda t}}{\lambda}\big)\exp\left(\int_0^{a\frac{1-e^{-\lambda t}}{\lambda}} \frac{\psi_z(x)}{a-\lambda x}dx\right) \tag{2.31}$$

Then we can compute the characteristic function of Y_t for different subordinators and we present here three examples that one can find in Carr et al. (2003) for which we recall below the characteristic functions:

a) For a process with Poisson arrival rate ν of positive jumps exponentially distributed with mean μ, we have a Lévy density that is

$$k_z(x) = \frac{\nu}{\mu}e^{-\frac{x}{\mu}}\mathbf{1}_{\{x>0\}}$$

and a log characteristic function

$$\psi_z(x) = \frac{ix\nu\mu}{1-ix\mu}$$

then we obtain

$$\int \frac{\psi_z(x)}{a-\lambda x}dx = \log\left(\big(x+\frac{i}{\mu}\big)^{\frac{\nu}{\lambda-i\mu a}}(a-\lambda x)^{\frac{\nu a\mu}{\lambda a\mu+i\lambda}}\right) \tag{2.32}$$

b) Let us consider the first time a Brownian motion with drift ν reaches 1. It is well known that this passage time follows the so-called Inverse Gaussian law which Lévy density and log characteristic function are respectively

$$\begin{aligned}k_z(x) &= \frac{e^{-\frac{\nu^2 x}{2}}}{\sqrt{2\pi x^3}}\mathbf{1}_{\{x>0\}} \\ \psi_z(x) &= \nu-\sqrt{\nu^2-2ix}\end{aligned}$$

and we then get

$$\begin{aligned}\int \frac{\psi_z(x)}{a-\lambda x}dx &= \frac{2\sqrt{\nu^2-2ix}}{\lambda}+\frac{2\sqrt{\nu^2\lambda-2ia}}{\lambda^{3/2}}\operatorname{arctanh}\left(\sqrt{\frac{\lambda(\nu^2-2ix)}{\nu^2\lambda-2ia}}\right) \\ &\quad -\frac{\nu\log(a-\lambda x)}{\lambda}\end{aligned}$$

c) Finally, recall the Stationary Inverse Gaussian case which Lévy density and log characteristic function are

$$\begin{aligned} k_z(x) &= \frac{(1+\nu^2 x)e^{-\frac{\nu^2 x}{2}}}{2\sqrt{2\pi x^3}}\mathbf{1}_{\{x>0\}} \\ \psi_z(x) &= \frac{iu}{\sqrt{\nu^2-2ix}} \end{aligned}$$

From these definitions, we obtain

$$\int \frac{\psi_z(x)}{a-\lambda x}dx = \frac{\sqrt{\nu^2-2ix}}{\lambda} - \frac{2ia}{\lambda^{3/2}\sqrt{\nu^2\lambda-2ia}}\operatorname{arctanh}\left(\sqrt{\frac{\lambda(\nu^2-2ix)}{\nu^2\lambda-2ia}}\right)$$

2.5 Correlation Adjustment

2.5.1 Introducing some Correlation

We propose a time-changed Bessel process as in the previous section with some leverage in order to get more independence between skewness and credit spreads, with respect to which we add a term that contains a negative correlation (equal to ρ) component between the stock return and the volatility. Hence, let us consider z_t a $\sigma(h_s, s \leq t)$ adapted positive integrable process such as

$$\frac{e^{\rho z_t}}{\mathbb{E}[e^{\rho z_t}]}$$

is a martingale and a general integrated time change $H_t = \int_0^t h_s ds$ such as $\mathbb{E}(H_t) < \infty$ then, we can define the stock price process as follows

$$\begin{aligned} S_t &= e^{rt} BESQ^{2-\delta}_{H_{t\wedge\tau}} \frac{e^{\rho z_t}}{\mathbb{E}[e^{\rho z_t}]} \\ \tau &= \inf\{t>0; S_t = 0\} \end{aligned}$$

where $BESQ$ is a squared Bessel process of dimension $\delta < 2$ starting from $S_0^{1/(2-\delta)}$.

Let us first show that the process $(e^{-rt}S_t; t \geq 0)$ hence defined is a martingale. We know from Proposition 2.4.1 that $BESQ^{2-\delta}_{H_{t\wedge\tau}}$ is a martingale. Now because of the independence of the processes z and $BESQ$

$$< BESQ^{2-\delta}_{H_{\cdot\wedge\tau}}, \frac{e^{\rho z_\cdot}}{\mathbb{E}[e^{\rho z_\cdot}]} >_t = 0$$

which ensures that $(e^{-rt}S_t; t \geq 0)$ is a local martingale. Let us show that it is actually a true martingale. For this purpose, let us recall some results:

Definition 2.5.1 *A real valued process X is of class DL if for every $a > 0$, the family of random variables $X_T\mathbf{1}_{\{T<a\}}$ is uniformly integrable for all stopping times.*

We now state the following property:

Proposition 2.5.2 *Let M_t be a local martingale such that $\mathbb{E}|M_0| < \infty$ and such that its negative part belongs to class DL. Then its negative part is a super-martingale. M_t is a martingale if and only if $\mathbb{E}[M_t] = \mathbb{E}[M_0]$ for all $t > 0$.*

Proof. The proof may be found in Elworthy, Li and Yor (1999). ■

All the financial assets being positive, one may use a simpler result than the previous property the proof of which is left to the reader:

Corollary 2.5.3 *Let M_t be a positive local martingale such that $\mathbb{E}|M_0| < \infty$. Then M_t is a super-martingale and it is a martingale if and only if $\mathbb{E}[M_t] = \mathbb{E}[M_0]$ for all $t > 0$.*

Consequently to prove that the actualized stock price process is a martingale with regards to the filtration $\mathcal{F}_t = \mathcal{R}_{H_t} \vee \sigma(h_s; s \leq t)$, we just need to show that for any $t > 0$

$$\mathbb{E}[e^{-rt}S_t] = S_0$$

which is the case since

$$\begin{aligned}
\mathbb{E}[e^{-rt}S_t] &= \mathbb{E}[BESQ^{2-\delta}_{H_{t\wedge\tau}}\frac{e^{\rho z_t}}{\mathbb{E}[e^{\rho z_t}]}] = \mathbb{E}\Big[\mathbb{E}\big[BESQ^{2-\delta}_{H_{t\wedge\tau}}\frac{e^{\rho z_t}}{\mathbb{E}[e^{\rho z_t}]}\big|\sigma(h_s; s \leq t)\big]\Big] \\
&= \mathbb{E}\Big[\frac{e^{\rho z_t}}{\mathbb{E}[e^{\rho z_t}]}\mathbb{E}\big[BESQ^{2-\delta}_{H_{t\wedge\tau}}\big|\sigma(h_s; s \leq t)\big]\Big] = \mathbb{E}\big[\frac{e^{\rho z_t}}{\mathbb{E}[e^{\rho z_t}]}S_0\big] = S_0
\end{aligned}$$

2.5.2 Pricing Credit and Equity Derivatives

The computation of the probability of default is immediate from Proposition 2.4.3 because

$$\tau = \inf\{t \geq 0; S_t = 0\} = \inf\{t \geq 0; BESQ_{H_t} = 0\}$$

and then for any $T > 0$

$$\mathbb{P}(\tau \leq T) = \mathbb{E}\Big[G(1 - \frac{\delta}{2}, \frac{S_0^{\frac{2}{2-\delta}}}{H_T})\Big]$$

where G is the complementary Gamma function.

Let us compute the European vanilla option prices. For this purpose, we define $C_0^{(\rho)}(x, y, \delta, K, T; S_0)$ and $P_0^{(\rho)}(x, y, \delta, K, T; S_0)$:

$$\begin{aligned}
C_0^{(\rho)}(x, y, \delta, K, T; S_0) &= S_0\frac{e^{\rho z_T}}{\mathbb{E}[e^{\rho z_T}]}Q(\frac{(Ke^{-(rT+\rho y)}\mathbb{E}[e^{\rho z_T}])^{\frac{2}{2-\delta}}}{x}, 4-\delta, \frac{S_0^{\frac{2}{2-\delta}}}{x}) \\
&\quad -Ke^{-rT}(1 - Q(\frac{S_0^{\frac{2}{2-\delta}}}{x}, 2-\delta, \frac{(Ke^{-(rT+\rho y)}\mathbb{E}[e^{\rho z_T}])^{\frac{2}{2-\delta}}}{x})) \\
P_0^{(\rho)}(x, y, \delta, K, T; S_0) &= Ke^{-rT}Q(\frac{S_0^{\frac{2}{2-\delta}}}{x}, 2-\delta, \frac{(Ke^{-(rT+\rho y)}\mathbb{E}[e^{\rho z_T}])^{\frac{2}{2-\delta}}}{x}) \\
&\quad -S_0\frac{e^{\rho z_T}}{\mathbb{E}[e^{\rho z_T}]}(1 - Q(\frac{(Ke^{-(rT+\rho y)}\mathbb{E}[e^{\rho z_T}])^{\frac{2}{2-\delta}}}{x}, 4-\delta, \frac{S_0^{\frac{2}{2-\delta}}}{x}))
\end{aligned}$$

Then, the knowledge of the joint law μ_{H_t,z_t} for any $t > 0$ enables us to compute the option prices as in the previous section:

$$\begin{aligned} C_0 &= \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} C_0^{(\rho)}(x,y,\delta,K,T;S_0)\mu_{H_t,z_t}(dx,dy) \\ P_0 &= \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} P_0^{(\rho)}(x,y,\delta,K,T;S_0)\mu_{H_t,z_t}(dx,dy) \end{aligned}$$

2.5.3 Examples

Let us go through most of the time changes presented previously and see how we can obtain the joint law of the couple (H_t, z_t).

Integrated CIR Time change Let us consider the following dynamics

$$dh_t = \kappa(\theta - h_t)dt + \eta\sqrt{h_t}dW_t^H$$

where W^H and $BESQ$ are independent and the stability condition $\frac{2\kappa\theta}{\eta^2} > 1$ is satisfied. Let us take

$$z_t = h_t + (\kappa - \frac{\rho\eta^2}{2})H_t$$

or equivalently

$$\rho z_t = \rho(h_0 + \kappa\theta t) + \rho\eta\int_0^t \sqrt{h_s}dW_s^H - \frac{\rho\eta^2}{2}\int_0^t h_s ds$$

Hence, it is obvious that

$$\frac{e^{\rho z_t}}{\mathbb{E}[e^{\rho z_t}]}$$

is a local martingale and it is known that it is a martingale as one may check using the Laplace transform below, that

$$\mathbb{E}[\exp(\rho\eta\int_0^t \sqrt{h_s}dW_s^H - \frac{\rho\eta^2}{2}\int_0^t h_s ds)] = 1$$

In order to compute credit and equity derivatives prices, we then compute for any positive λ, μ the Laplace transform of

$$\mathbb{E}[e^{-\lambda H_t - \mu h_t}]$$

It is well known (see Karatzas and Shreve (1991) or Lamberton and Lapeyre (1995)) that

$$\begin{aligned} \mathbb{E}[e^{-\lambda H_t - \mu h_t}] &= \frac{e^{\frac{\kappa^2\theta t}{\eta^2}}}{\big(\cosh(\gamma t/2) + \frac{\kappa+\mu\eta^2}{\gamma}\sinh(\gamma t/2)\big)^{2\kappa\theta/\eta^2}} \exp\big(-h_0 B(t,\lambda,\mu)\big) \\ B(t,\lambda,\mu) &= \frac{\mu\big(\gamma\cosh(\frac{\gamma t}{2}) - \kappa\sinh(\frac{\gamma t}{2})\big) + 2\lambda\sinh(\frac{\gamma t}{2})}{\gamma\cosh(\frac{\gamma t}{2}) + (\kappa + \lambda\eta^2)\sinh(\frac{\gamma t}{2})} \\ \gamma &= \sqrt{\kappa^2 + 2\eta^2\lambda} \end{aligned}$$

Heston CESV with correlation In the same class of models, let us now construct z in terms of the solution of the following stochastic differential equation

$$d\sigma_t^2 = \kappa(\theta - \sigma_t^2)dt + \eta\sigma_t dW_t^H \quad \text{and} \quad \sigma_0^2 = x.$$

First, h is defined by

$$h_t = \frac{\sigma_t^2 e^{2rt/(\delta-2)}}{(2-\delta)^2}$$

Then, following the same method as in the integrated CIR time change case, we choose z as:

$$z_t = h_t + (\kappa - \frac{2r}{\delta-2} - \frac{\rho\eta^2}{2})H_t$$

Consequently, $\frac{e^{\rho z_t}}{\mathbb{E}[e^{\rho z_t}]}$ is a martingale.

Hence, it remains to evaluate for any positive t the Laplace transform of (H_t, z_t), that is to say for any positive λ, μ

$$\mathbb{E}[e^{-\lambda H_t - \mu h_t}]$$

In order to compute the above quantity, we use the following result which extends Lemma 2.4.6 that one can find in Pitman and Yor (1982).

Lemma 2.5.4 *If X a squared Bessel process $BESQ_{(\delta,x)}$ starting from $x \neq 0$ and of dimension δ, then for any functions f and $g : \mathbb{R}_+ \to \mathbb{R}_+$ such that for any $t > 0$: $\int_0^t f(s)ds < \infty$, we have*

$$\mathbb{E}\Big[\exp\big(-\int_0^t X_s f(s)ds - g(t)X_t\big)\Big] = \frac{1}{(\psi_f'(t) + 2g(t)\psi_f(t))^{\delta/2}} \times \exp\frac{x}{2}\left(\phi_f'(0) - \frac{\phi_f'(t) + 2g(t)\phi_f(t)}{\psi_f'(t) + 2g(t)\psi_f(t)}\right)$$

where ϕ_f is the unique solution of the Sturm-Liouville equation

$$\phi_f''(s) = 2f(s)\phi_f(s)$$

where $s \in [0;\infty[$, $\phi_f(0) = 1$, ϕ_f is positive and non-increasing and

$$\psi_f(t) = \phi_f(t)\int_0^t \frac{ds}{\phi_f^2(s)}$$

Taking

$$g(t) = \frac{\mu}{(2-\delta)^2}\left(\frac{\eta^2}{4\kappa t + \eta^2}\right)^{1+\frac{2r}{\kappa(2-\delta)}}$$

in the above Lemma, we obtain that for any positive λ, μ

$$\mathbb{E}[e^{-\lambda H_t - \mu h_t}] = \frac{1}{(\psi_\lambda'(l(t)) + 2\frac{\mu}{(2-\delta)^2}e^{-(\kappa+\frac{2r}{2-\delta})t}\psi_\lambda(l(t)))^{\delta/2}} \times \exp\frac{x}{2}\left(\phi_\lambda'(0) - \frac{\phi_\lambda'(l(t)) + 2\frac{\mu}{(2-\delta)^2}e^{-(\kappa+\frac{2r}{2-\delta})t}\phi_\lambda(l(t))}{\psi_\lambda'(l(t)) + 2\frac{\mu}{(2-\delta)^2}e^{-(\kappa+\frac{2r}{2-\delta})t}\psi_\lambda(l(t))}\right)$$

where noting $\alpha = \frac{\delta-1}{\delta-2}$, the functions ϕ_λ, ψ_λ and l are defined respectively in (2.25),(2.26) and (2.27).

Integrated Ornstein-Uhlenbeck Time Change We consider the stochastic time change $H_t = \int_0^t h_s ds$ and assume that $(h_t; t \geq 0)$ is given by

$$dh_t = -\lambda h_t dt + dz_t$$

where $(z_t; t \geq 0)$ is a Lévy subordinator. Carr et al. (2003) compute the characteristic function $\Phi(t, a, b)$ of (H_t, z_t) for any $t > 0$ and it is given by

$$\mathbb{E}[e^{iaH_t + ibz_t}] = \exp\big(iah_0 \frac{1-e^{-\lambda t}}{\lambda}\big) \exp\Big(\int_b^{b+a\frac{1-e^{-\lambda t}}{\lambda}} \frac{\psi_z(x)}{a+\lambda b - \lambda x} dx\Big) \tag{2.33}$$

for any $(a, b) \in \mathbb{R}_+^2$ where ψ_z is the log characteristic function of the subordinator. Let us first notice that

$$\mathbb{E}[e^{\rho z_t}] = \exp(t\psi_z(-i\rho))$$

We quickly recall the computations of $\Phi(t, a, b)$ for different subordinators:
a) For a process with Poisson arrival rate ν of positive jumps exponentially distributed with mean μ, we obtain

$$\int \frac{\psi_z(x)}{a+\lambda b - \lambda x} dx = \log\Big(\big(x + \frac{i}{\mu}\big)^{\frac{\nu}{\lambda - i\mu(a+\lambda b)}} ((a+\lambda b) - \lambda x)^{\frac{\nu(a+\lambda b)\mu}{\lambda(a+\lambda b)\mu + i\lambda}}\Big)$$

b) For an Inverse Gaussian subordinator of parameter ν, we have

$$\begin{aligned}\int \frac{\psi_z(x)}{a+\lambda b - \lambda x} dx &= \frac{2\sqrt{\nu^2 - 2ix}}{\lambda} \\ &+ \frac{2\sqrt{\nu^2\lambda - 2i(a+\lambda b)}}{\lambda^{3/2}} \operatorname{arctanh}\Big(\sqrt{\frac{\lambda(\nu^2 - 2ix)}{\nu^2\lambda - 2i(a+\lambda b)}}\Big) \\ &- \frac{\nu \log((a+\lambda b) - \lambda x)}{\lambda}\end{aligned}$$

c) For the Stationary Inverse Gaussian of parameter ν, we write

$$\begin{aligned}\int \frac{\psi_z(x)}{a+\lambda b - \lambda x} dx &= \frac{\sqrt{\nu^2 - 2ix}}{\lambda} \\ &- \frac{2i(a+\lambda b)}{\lambda^{3/2}\sqrt{\nu^2\lambda - 2i(a+\lambda b)}} \operatorname{arctanh}\Big(\sqrt{\frac{\lambda(\nu^2 - 2ix)}{\nu^2\lambda - 2i(a+\lambda b)}}\Big)\end{aligned}$$

2.6 Conclusion

Twelve continuous stochastic stock price models were built in this paper for equity-credit modelling purposes, all derived from the Constant Elasticity of Variance model, and as a consequence from Bessel processes. They all exploit the ability of Bessel processes to be positive, for those of dimension lower than 2 to reach 0 and for a certain power of a given Bessel process to be a martingale. We first propose to add a stochastic volatility diffusion to the CEV model, then more generally to stochastically time change a Bessel process in order to obtain a stochastic volatility effect, motivated by known arguments that go back to Clark (1973). Next, in order to add some correlation between the stock price process and the stochastic volatility, we extend our framework by multiplying the Bessel process by exponentials of the volatility and correcting it by its mean in accordance with arbitrage considerations to obtain martingale models that are martingales with respect to the joint filtration of the time-changed Bessel process and the stochastic time change itself. Hence, among the different models proposed based on the CEV with default model, there were first the Constant Elasticity of Stochastic Variance ones (CESV) taking a Hull and White (1987) stochastic volatility as well as a Heston (1993) one. We then proposed integrated time change models, by considering an integrated CIR time change and an Integrated Ornstein-Uhlenbeck time change (see Carr et al. (2003)) with different subordinators for the process driving the diffusion. We finally added correlation between stock price returns and volatilities to the models presented previously and provided quasi-analytical formulas for option and CDS prices for all of them. Let us note that we discussed the true and local strict martingale properties of CEV processes, that we naturally extended to the time change framework.
The models presented and discussed in this paper are not specifically designed to cope just with Equity-Credit frameworks but they also can be used for instance for FX-rates hybrid modelling by specifying stochastic interest rates. We can also note that a Poisson jump to default process can be added to the CEV-like framework in order to deal with credit spreads for short-term maturities. Campi, Polbennikov and Sbuelz (2005) and Carr and Linetsky (2005) precisely considered a CEV model with deterministic volatilities and hazard rates. The latest paper can easily be generalized to fit in our time-changed Bessel frameworks. Since our goal was to concentrate on continuous diffusions, we leave the addition of a jump to default for further research.

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Chapter 3

Hybrid Equity-Credit Modelling

[*Joint work with Boris Leblanc; published in Risk Magazine, August 2005**]

We propose a study of the pitfalls of the market widely used Poisson Default model in the Equity-Credit Hybrid land and show that a slight modification of the Constant Elasticity of Variance (CEV) model can, in addition to its well-known properties, capture the default event probability. Because of a need of more freedom between the volatility level, the skewness and the risk of default, we exhibit extensions of the CEV model adding stochasticity in the volatility.

3.1 Introduction

The growth of the credit derivatives market and the development of derivatives such as equity default swaps (EDSs) has led to a need for models that realistically capture stock price behaviour. The probability of default has become a crucial issue for pricing new claims. We therefore need to define "default".

The notion of default has been discussed in market financial literature for a long time and in corporate finance literature for much longer. Defaults happen when a party is unwilling or unable to pay its debt obligations. Default is usually the step before bankruptcy in corporate finance. In the US, a firm getting in trouble usually files for bankruptcy protection under Chapter 11 (Reorganization), which defines a default event. Chapter 11 allows a firm to cancel some or all of its debts and contracts while attempting to achieve financial stability without interruption of the operating business.

In the common structural model literature pioneered by Black & Scholes (1973) and Merton (1974), one defines default as being the event for which the asset value of a firm goes below a boundary that is a function of the firm debts. But the impossibility of knowing the barrier level leads us to consider alternatives to structural models. Reduced-form models do not model the value of the firm's assets and its capital structure, they consider the credit event to be an exogenously specified jump process. Two reduced-form model subclasses are the credit migration model family

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and the intensity-based model family. In the case of intensity- based models, one is interested in modelling the default event process; the traditional literature on these kinds of models (see, for instance, Jarrow and Turnbull (1995)) does not describe the behaviour of stocks just before default.

Since our aim is to build a unique model for stock prices and default events, modelling the probability of default as the consequence of the stock price falling under a certain boundary seems natural. Now, for simplicity, recent models (see, for example, Albanese and Chen (2005) and Linetsky (2005)) consider the default event as the stock price falling to zero and this is the framework we will use. We may nevertheless notice that for a firm, the fact of being under Chapter 11 doesn't imply that the stock price is equal to zero, but being under Chapter 7 (Liquidation) will imply a null stock price. As lognormal models are unable to comply with this latest feature, financial practitioners and academics have added to the diffusion model a Poisson default process. Such models were first presented in Davis and Lischka (2002), where the default probability depends on the level of the spot price. In this article, we wish to build a stock price diffusion with continuous paths, since for most companies going bankrupt the stock price behaviour doesn't default as a Poisson process does. This stock price property can be illustrated on the US stock market (see for instance WorldCom, Enron, Mirant or Kmart) and that is the reason why building continuous processes with a non-zero probability of reaching zero is a very interesting feature.

The constant elasticity of variance (CEV) model designed by Cox (1975) is a continuous path model that has the following diffusion $\frac{dS}{S} = rdt + \sigma S^{\alpha-1} dW$ and a non-zero probability of reaching zero under certain conditions on the elasticity parameter α.

First we will explain why the CEV model describes the equity market better than the Poisson default model in terms of realism of the stock price paths and pricing downside risks. By this we mean that the path continuity of the CEV model brings consistency with low-strike put options and equity default swap (EDS) market prices, for example. We will then more precisely present the stopped CEV process and price vanilla options, credit default swaps (CDSs) and equity default swaps within this model. The major drawbacks of the CEV model are the lack of independence between the skewness and the probability of default, and the high dependency between the level of volatility and the probability of default. To deal with these drawbacks, we will present some generalizations of the CEV model using stochastic volatility. Our contribution is threefold: explaining default Poisson model mis-pricing features and illustrating the necessity of smoother stock price processes (with continuous paths) to model the default event; showing that the stopped CEV model can approximately fit vanilla options and CDSs and price EDSs more safely; introducing and presenting an extension of the stopped CEV model using Heston stochastic volatility (constant elasticity of stochastic variance (CESV)). We additionally provide closed-form pricing formulas for the Heston CESV model presented in this article.

3.2 Tracking a Stock Price Process that models default

To price exotic derivatives, it is first necessary to be able to reproduce existing, observable vanilla option prices with sufficient precision and a small number of adjustable parameters. The main drawback of this view is often the irrelevance of the underlying asset price behaviour and, as a consequence, a lack of accuracy for the hedging portfolio. The local probability of default model doesn't represent a typical path of a default event since the stock price process can jump to zero at any time with a probability that is a function of the underlying stock price. Our purpose is to create a model consistent with the sustainable stock price evolutions. An important feature of a stochastic model is its ability to integrate extremal events realistically. When building a model concerned with default, the choice of the diffusion may be of importance for pricing non plain-vanilla derivatives such as EDSs that are swaps where payouts occur when the stock price falls under a pre-defined level. We will now recall the Poisson default model and present a slight modification of the CEV model as an alternative to the unrealistic stock behaviour of the Poisson default process.

3.2.1 Poisson Default Models

The most commonly used equity-credit market models are those based on jump-diffusion processes with a jump to zero if the stock defaults. This type of model usually solves the following equation under the risk-neutral measure:

$$\frac{dS_t}{S_{t-}} = rdt + \sigma dW_t - dQ_t$$

where:

$$\tau = \inf\{t > 0; \int_0^t p(u, S_u)du \geq \Theta\}$$
$$Q_t = \mathbf{1}_{t \geq \tau} - \int_0^{t \wedge \tau} p(u, S_u)du$$

where Θ is an exponential random variable, p is a deterministic function of the time and the spot level. This model was presented for instance in Davis and Lischka (2002) and is commonly used for the pricing of defaultable claims, especially of convertible bonds. In Andersen and Buffum (2003) and in Ayache et al. (2003) for instance, the probability function is of the following form:

$$p(S) = p_0(\frac{S}{S_0})^{\alpha}$$

where p_0 is the estimated hazard rate for the stock price level $S = S_0$. Linetsky (2005) provides closed-form formulae for vanilla option prices and corporate bonds with the specification on the local probability function presented above.

Such processes generate paths where the stock price drops down directly to zero from its level just before default. As shown in figure 1, it is not a natural hypothesis for a default modelling framework and that is why we consider alternative smoother processes.

3.2.2 CEV Diffusion

A positive continuous process that has a strictly positive probability of reaching zero can be found in the family of squared Bessel processes with dimensions lower than two. Among the different stock price models, the CEV model is a well-known stock price diffusion based on Bessel processes. In this article, we will consider a CEV process stopped at the first hitting time of zero in order to build a credit-coherent model under a risk-neutral pricing measure:

$$\begin{aligned} \frac{dS_t}{S_t} &= rdt + \sigma S_t^{\alpha-1} dW_t \quad if \quad t < \tau. \\ S_t &= 0 \quad if \quad t \geq \tau. \end{aligned}$$

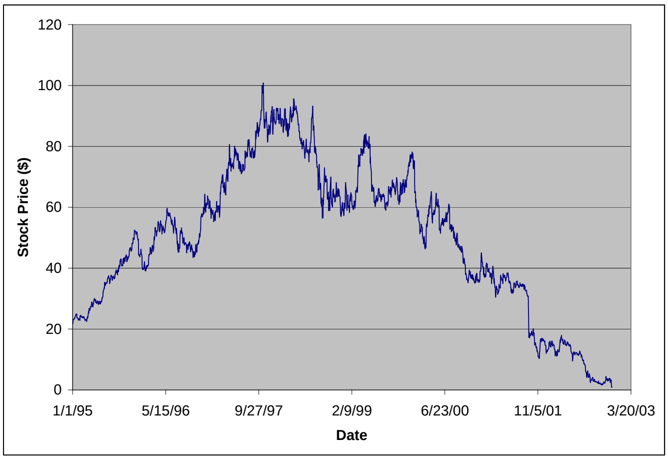

Figure 3.1: United Airlines Historical Stock Price prior to Default

where $\tau = T_0(S) = \inf\{t > 0, S_t = 0\}$, σ a constant and $\alpha < 1$ in order to get a non-zero probability of default.

For a CEV process, zero is an absorbing boundary for $\frac{1}{2} < \alpha < 1$ and is a reflecting boundary for $\alpha < \frac{1}{2}$ and this is why we consider a stopped CEV process. There is much literature on CEV models. Since CEV processes are based on squared Bessel processes, they have the advantage of giving analytical formulas for many derivatives. They were introduced by Cox (1975), who only considered the case $\alpha < 1$, which takes into account the so-called leverage effect to price vanilla options. Then Emanuel and Mac-Beth (1982) proposed pricing formulas for $\alpha > 1$ and Schroder (1989) showed that the CEV pricing formula could be expressed in terms of noncentral chi-square distributions. More recently, Delbaen and Shirakawa (2002) proposed a call pricing formula for the stopped CEV process. We can calculate the law of this stopped process in terms of squared Bessel processes and hence in terms of non-central chi-square distributions. For a detailed study of the stopped process, we refer to Delbaen and Shirakawa (2002). But, for the purpose of self-consistency, some essential results are reproduced in the Appendix.

3.2.3 Poisson Default Process Problem

We aim at continuous diffusions that can reach zero. A possible inconsistency of Poisson default models comes from the pricing of EDSs. If we wish to price an EDS with a low implied volatility and a high credit grade, there won't be a significant price difference between a quarterly 20% two-year EDS and a quarterly 30% two-year EDS. For example, let us consider the US company Tyco, with a 23% one-year at-the-money implied volatility and a 250-basis point one-year credit grade with a \$36.50 spot price. For simplicity, we will consider down-and-in barrier put options whose payout is of the following form: $\mathbb{E}[e^{-r\tau}\mathbf{1}_{\tau<T}]$ with $\tau = \inf_{t<T}\{S_t < B\}$ and calculate their prices under the CEV model and under the Poisson default model. All the prices can be found in table A. The two models are fitted on the one-year at-the-money volatility and probability of default. The prices under the Poisson default model show a bad strike scaling feature. We see that reaching low barriers is equivalent to reaching zero in our jumpdiffusion framework for the pricing of down-and-in barrier put options, and that is why they all have the same price. In the CEV model, they all have different prices and they are more expensive than in the jump-diffusion model. Let us remark that to get the one-year Poisson default model price presented in table A, one would need to take a 5% barrier to get the same price under the CEV model. Nonetheless, one could argue that the default event could be chosen not to be zero but a certain small value ê as presented in its generality in Ayache, Forsyth and Vetzal (2003), but that would only shift up the options prices and they would remain insensitive to strike scaling. Another explanation of this important price difference can be excerpted from a qualitative study of the hedging strategy, and this will illustrate another problem with Poisson default models. Indeed, when selling a down-and-in digital put barrier option under a Poisson default model, in the case of a jump to zero the profit will come from the number of short stocks. But the stock price usually declines smoothly before jumping down in case of default, so the Poisson default model won't perform efficiently, whereas managing these options under the CEV model is better because the intrinsic structure of this model sees the default event as it may happen. This means that for the pricing involved, the delta for the CEV process is higher than for the default Poisson process. To summarise, these price differences come from different hedging strategies, which themselves come from different stock price behaviour modelling.

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DOWN-AND-IN DIGITAL BARRIER OPTION PRICES IN DOLLARS

Poisson Default Model, $\sigma = 20.2\%$, $\alpha = 2$, $r = 2\%$ and $p_0 = 3.7$

Strike/Maturity	1 Year
30%	0.0246
40%	0.0246
50%	0.0249

CEV Model, $\sigma = 23\%$, $\alpha = -1.6$ and $r = 2\%$

Strike/Maturity	1 Year
30%	0.045
40%	0.062
50%	0.083

3.3 Consistent Pricing of Credit and Equity Derivatives within CEV

3.3.1 Calibration and Pricing of Vanilla Options

Since our purpose is to build a cross-asset market model for strategies that involve equity and credit assets, we calculate the European-style vanilla option prices. To ensure the absence of arbitrage, the discounted stopped CEV process has to be a true martingale. This is the case for $\alpha < 1$, as proven in Atlan and Leblanc (2004).

Let us now calculate the European-style put P_0 option price at maturity T and strike K for the stopped CEV process:

$$P_0 = e^{-rT}\mathbb{E}[(K - S_T)_+ \mathbf{1}_{T<\tau}] + Ke^{-rT}\mathbb{P}(\tau \leq T)$$

We can see explicitly that the put option price incorporates the price of default and that the martingale property ensures the put-call parity relation. We can now give the option pricing formula,

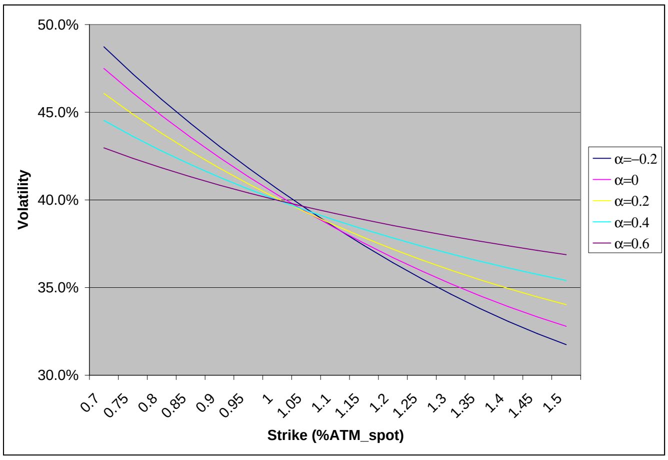

Figure 3.2: CEV Skewness, $\sigma = 40\%$, $r = 2\%$ and T=1 year

knowing the density of the stopped CEV process thanks to equation (3.11) that one can find in the Appendix:

$$P_0 = Ke^{-rT}Q(2\xi_T, \frac{1}{1-\alpha}, z_T) - S_0(1 - Q(z_T, 2 + \frac{1}{1-\alpha}, 2\xi_T))$$

where

$$\begin{aligned} z_T &= \frac{2rK^{2(1-\alpha)}}{\sigma^2(1-\alpha)(e^{2(1-\alpha)rT} - 1)} \\ \xi_T &= \frac{rS_0^{2(1-\alpha)}}{(1-\alpha)\sigma^2(1 - e^{-2(1-\alpha)rT})} \end{aligned}$$

and Q is the complementary non-central chi-square distribution function. One can obtain the call option price thanks to the Call-Put parity relation. For homogeneity reasons, we may define σ_0 to be such as:

$$\sigma = \frac{\sigma_0}{S_0^{\alpha-1}}$$

To fit an implied volatility curve at a given maturity, take :

$$\begin{aligned} \sigma_0 &\simeq \sigma_{ATM}^{BS} \\ \frac{\partial \sigma^{BS}}{\partial K} &\simeq \frac{\sigma_0(\alpha - 1)}{S} \end{aligned}$$

These approximations enable us to get a good idea of the parameters. Figure 2 shows several skews generated by a CEV model at a given at-the-money implied volatility for a given maturity. Figure 3 shows General Motors' implied volatility skew for the maturity January 2006 as of May 2005. The calibration of α and σ_0 was performed for a given maturity on all the call options where bids and asks were provided.

Using the General Motors calibrated volatility curve, the calculated credit grade of the one year CDS with a recovery rate R = 30 is 326bp.

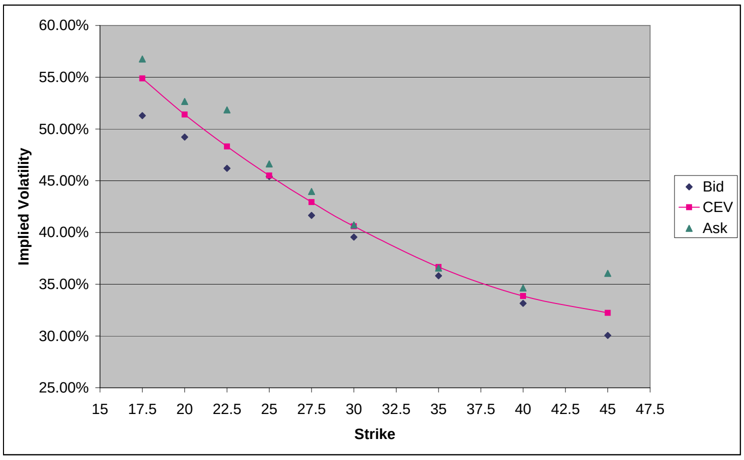

Figure 3.3: General Motors January 06 Implied Volatility Curve, $\sigma_0 = 43\%$, $\alpha = -0.28$, $S_0 = \$27$ and $r = 2\%$

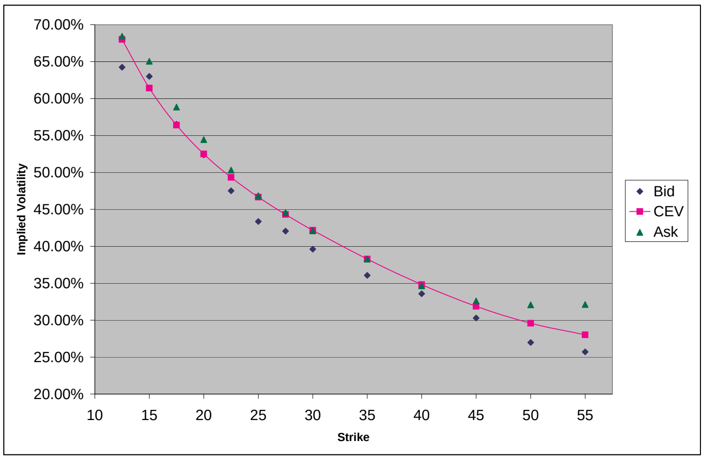

Figure 3.4: General Motors January 07 Implied Volatility Curve, $\sigma_0 = 41\%$, $\alpha = -0.28$, $S_0 = \$27$ and $r = 2\%$

Figure 4 displays the implied volatility curve for January 2007 call options based on the calibration of the á performed on January 2006 options and on an adjustment of σ_0 to the at-the-money volatility. It is well known that for short-term maturities, jumps are needed in the dynamic to perform a model calibration. That is the reason why adding a regular Poisson jump process to the CEV model allows a short-term maturity calibration. However, this is not the aim of this article and we leave it for further research.

3.3.2 Credit Derivatives Pricing

In the past few years, with the growth of the credit derivatives market, the issue of pricing CDSs with a view on the equity market has become important, especially with the recent interest in EDS pricing.
To calibrate a model to CDS market prices, we need to be able to calculate the probability of default in the CEV framework. That means we want to calculate the first hitting time of the zero cumulative distribution function. This calculation was originally done by Cox (1975). Not long afterwards, the cumulative distribution function was computed for Bessel processes by Getoor (1979). We obtain the following simple formula for the CEV process:

$$\mathbb{P}(\tau \leq T|S_0) = G(\frac{1}{2(1-\alpha)}, \xi_T) \tag{3.1}$$

where G and ξ_T are defined as follow:

$$\begin{aligned} G(x,y) &= \int_{z\geq y} \frac{z^{x-1}e^{-z}}{\Gamma(x)} \mathbf{1}_{\{z>0\}} dz \\ \xi_T &= \frac{rS_0^{2(1-\alpha)}}{(1-\alpha)\sigma^2(1-e^{2(\alpha-1)rT})} \end{aligned}$$

This last formula enables us to calibrate the CEV model to the CDS market. We recall the general valuation formula of a CDS initiated at time zero and evaluated at time t:

$$CDS_t(T_1, T_n; C; R) = -C\sum_{i=1}^{n} B(t, T_i)\mathbb{P}(\tau > T_i|S_t) + (1-R)\mathbb{E}[e^{-r(\tau-t)}\mathbf{1}_{\tau\leq T_n}|S_t]$$

where C is the coupon, $T_1, ..., T_n$ the payment dates, $B(t, T_i)$ the risk-free zero-coupon bonds, r the risk-free interest rate, R the recovery rate assumed to be deterministic, τ the default time and $\mathbb{P}(\tau > T_i|S_t)$ is given by formula (3.1). Figure 5 illustrates the different probabilities of default generated for a given level of at-the-money 1-year implied volatility within the CEV model. In the absence of arbitrage the coupon value at the inception of the contract is given by:

$$CDS_{t=0}(T_1, T_n; C; R) = 0$$

To price a CDS within the CEV Model, we just need to compute the rebate price that can be found in Davydov and Linetsky (2001).

EDSs are very similar to CDSs except that payouts occur when the stock price falls under a pre-defined level, which is often referred to as a trigger price. The trigger price is usually around 30% of the equity stock price at the beginning of the contract. Hence, these contracts provide a protection against a credit event happening on the equity market for the buyer. They were initiated by the end of 2003. At that time, it had become difficult in many countries to structure investment-grade credit portfolios with good returns because the CDS spreads were tightening, as reported by Sawyer (2003). Let us now define τ_L as the first passage of time of the stock price process under the level L < S0. Formally, we write $\tau_L = \inf\{t > 0; S_t \leq L\}$. We recall the general valuation formula of an EDS:

$$EDS_t(T_1, T_n; C; R) = -C\sum_{i=1}^{n} B(t, T_i)\mathbb{P}(\tau_L > T_i|S_t) + \mathbb{E}[e^{-r(\tau_L-t)}\mathbf{1}_{\tau_L\leq T_n}|S_t]$$

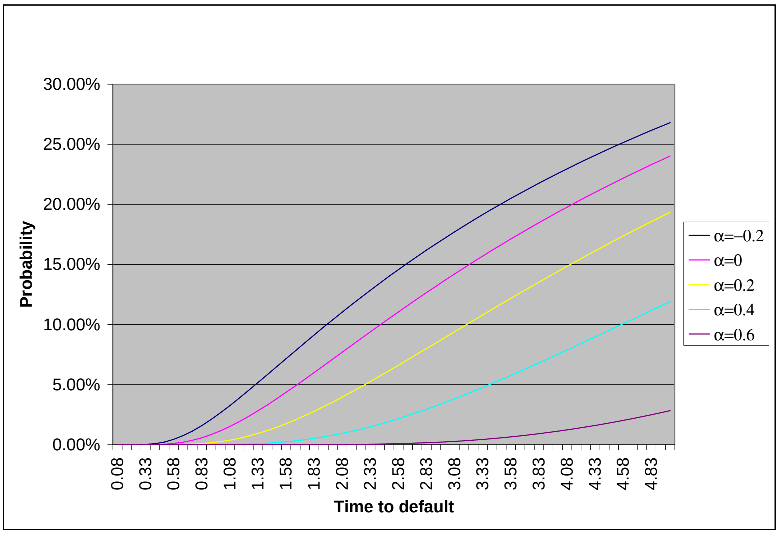

Figure 3.5: CEV Probabilities of default, $\sigma = 40\%$ and $r = 2\%$

where C is the coupon, $T_1, ..., T_n$ the payment dates, $B(t, T_i)$ the risk-free zero-coupon bonds and r the risk-free interest rate. Again, by absence of arbitrage, we can find the coupon price, by stating that at the initiation of the contract:

$$EDS_{t=0}(T_1, T_n; C; R) = 0$$

Analytical formulae for the EDS price are obtained using Davydov and Linetsky (2001) and can be found in Albanese and Chen (2004).

3.4 Heston CESV Model

Due to the limitations in the CEV model's ability to capture the main derivatives market effects - that is to say some flexibility between the level of volatility, the probability of default and the smile structure - we are led to consider a stochastic volatility instead of a constant one. More precisely, it enables us to cope with the bad time dependency of CEV credit curves for stocks with low volatilities and high probabilities of default. A well-known model of this family used in fixed income is the SABR model introduced by Hagan et al (2002). Hence, we wish to build a Heston stochastic volatility model with a CEV diffusion for the stock price dynamics:

$$\frac{dS_t}{S_t} = rdt + \sigma\sqrt{v_t}S_t^{\alpha-1}dW_t^S \qquad if \qquad t < \tau. \tag{3.2}$$

$$dv_t = \kappa(1 - v_t)dt + \eta\sqrt{v_t}dW_t \tag{3.3}$$

$$v_0 = 1 \tag{3.4}$$

$$S_t = 0 \qquad if \qquad t \geq \tau \tag{3.5}$$

$$d < W, W^S > = 0 \tag{3.6}$$

where W and W^S are standard Brownian motions, $\tau = T_0(S) = \inf\{t > 0, S_t = 0\}$ and $\alpha < 1$. We do not correlate the stock price return dynamics and the volatility process because the leverage effect is sufficiently well explained by the constant elasticity effect. Adding a stochastic volatility also permits the capture of a smile effect less correlated to the probability of default and changes of regimes in volatility that are shown in figure 6.

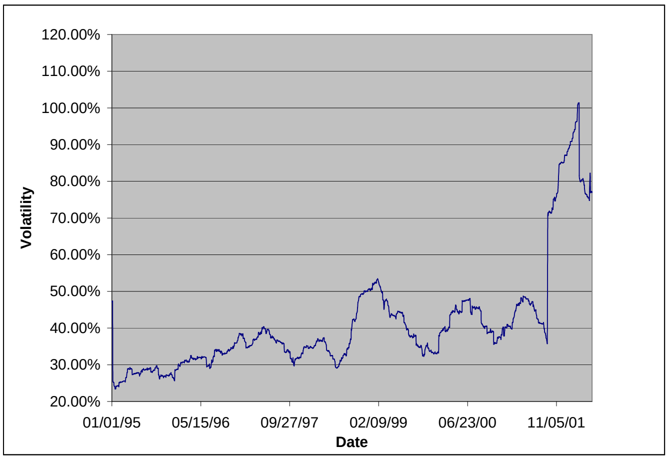

Figure 3.6: United Airlines 6 month Historical Volatility prior to Default

Therefore, if we consider the following process X defined as follows:

$$X_t = e^{rt} R^{\frac{1}{2(1-\alpha)}}_{(2-\frac{1}{1-\alpha}, x_0^{2(1-\alpha)})}(H_t) \qquad if \qquad t < \tau \tag{3.7}$$

$$X_t = 0 \qquad if \qquad \tau > t \tag{3.8}$$

where $H_t = \sigma^2(1-\alpha)^2 \int_0^t v_s e^{-2(1-\alpha)rs} ds$ and $R_{(\delta,x)}$ is a Squared Bessel Process, we can show that this process is a solution of Equations (3.2) and (3.5). To prove this relation, it suffices to apply Ito Formula and the change of variable formula. A crucial point in the use of stochastic volatility is that the absence of Arbitrage is expressed by the property of the discounted stock price process being a true martingale, as mentioned above. Now, using the conditioning formula, we are able to get formulae that just depend on the law of H at terminal time. More precisely, defining the following quantity $P_0(x, K, T; S_0)$ by:

$$\begin{aligned} P_0(x, K, T; S_0) &= Ke^{-rT} Q(\frac{S_0^{2(1-\alpha)}}{x}, \frac{1}{1-\alpha}, \frac{(Ke^{-rT})^{2(1-\alpha)}}{x}) \\ &\quad - S_0(1 - Q(\frac{(Ke^{-rT})^{2(1-\alpha)}}{x}, 2 + \frac{1}{1-\alpha}, \frac{S_0^{2(1-\alpha)}}{x})) \end{aligned}$$

we obtain the put option price:

$$P_0 = \int_{\mathbb{R}_+} P_0(x, K, T; S_0) \mu_{H_T}(dx)$$

where μ_{H_T} is the law of H at time T. The call option price may be obtained using the call-put parity relation.
The probability of default can still be different from 0 and we have:

$$\mathbb{P}(\tau \leq T) = \int_{\mathbb{R}_+} p(x; S_0)\mu_{H_T}(dx) \tag{3.9}$$

where $p(x, S_0) = G(\frac{1}{2(1-\alpha)}, \frac{S_0^{2(1-\alpha)}}{x})$.

It is well known that the law of this process can be expressed in terms of a space and time changed squared Bessel processes. A condition on v_t ensuring that 0 remains a reflecting boundary is $\frac{4\kappa}{\eta^2} > 0$ and a stability condition ensuring that the volatility process remains strictly positive is that $\frac{4\kappa}{\eta^2} > 2$. Let us for simplicity reasons consider a CEV diffusion for the forward contract F_t, it will then solve the SDE below:

$$\begin{aligned}
\frac{dF_t}{F_t} &= \sigma\sqrt{v_t}S_t^{\alpha-1}dW_t^F \quad if \quad t < \tau \\
F_t &= 0 \quad if \quad t \geq \tau \\
dv_t &= \kappa(1-v_t)dt + \eta\sqrt{v_t}dW_t \\
v_0 &= 1 \\
d < W, W^F > &= 0
\end{aligned}$$

where W^F is a brownian motion and $\tau = T_0(F) = \inf\{t > 0, F_t = 0\}$.

Consequently, we are looking for the law of:

$$H_t = \sigma^2(1-\alpha)^2 \int^t v_s ds \tag{3.10}$$

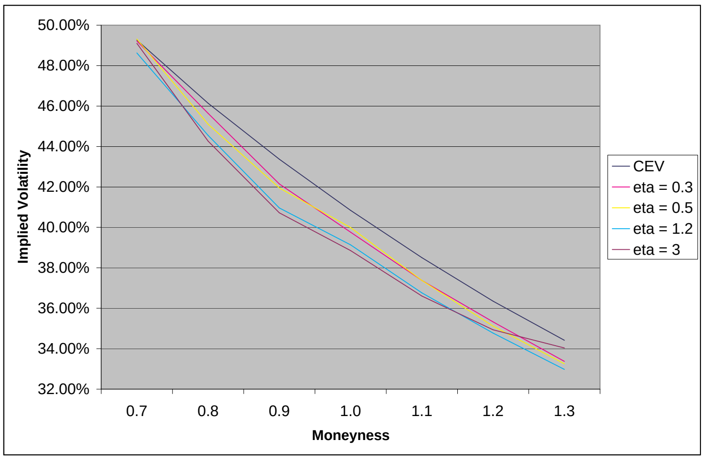

Figure 3.7: Heston CEV Skewness, $\sigma_0 = 40\%, \kappa = 2, \alpha = -0.3$, $r = 2\%$ and T=1 year

Hence, we are able to compute the Laplace transform of H_t and then get the law of H_t. It is a well-known computation for those who are for example, calculating the price of discount bonds within a CIR (1985) model. Let us recall its Laplace transform that one can find for instance, in Lamberton and Lapeyre (1995) $\forall \lambda \in \mathbb{R}_+$:

$$\mathbb{E}[e^{-\lambda H_t}] = e^{-\kappa \varphi_\lambda(t)} e^{-\psi_\lambda(t)}$$

where:

$$\begin{aligned}
\gamma &= \sqrt{\kappa^2 + 2\eta^2 \lambda \sigma^2 (1-\alpha)^2} \\
\varphi_\lambda(t) &= -\frac{2}{\eta^2} \ln \left(\frac{2\gamma e^{t\frac{\gamma+\kappa}{2}}}{\gamma - \kappa + e^{\gamma t}(\gamma+\kappa)} \right) \\
\psi_\lambda(t) &= \frac{2\lambda\sigma^2(1-\alpha)^2(e^{\gamma t}-1)}{\gamma - \kappa + e^{\gamma t}(\gamma+\kappa)}
\end{aligned}$$

Figure 7 shows the impact of the addition of a stochastic volatility to the smile structure. One can see that a simple way to get an upward smile for upside strikes is to take κ and η such that $1 > \frac{2\kappa}{\eta^2} > 0$.

3.5 Conclusion

This article presents a study of the CEV model and an analysis of one of its possible extensions where we add a stochastic volatility (CESV model), both dedicated to the pricing of credit derivatives and equity derivatives where a downside risk is involved. We have shown that the widely used Poisson default model cannot represent the stock price behaviour of a firm defaulting, and thus a process is needed with a continuous component that by itself can "easily" reach low spot levels. This is the case of the well-known CEV model, and that is why we considered a slight modification involving stopping the CEV process at its first-passage time by zero, to be consistent with the default event. Then, to get more freedom in the correlation structure of the skewness with the level of default, we naturally build CESV models. Moreover, for some stochastic volatility models, we are able to calculate analytical formulas. At this point, we note that we haven't performed any hedging strategies based on the CEV-type models. We leave this for future research. We also leave for future research the study of models mixing jumps and diffusions able to reach zero, such as a Poisson default CEV model that would solve the following SDE:

$$\frac{dS_t}{S_{t-}} = rdt + \sigma S_{t-}^{\alpha-1} dW_t - dQ_t$$

where :

$$Q_t = \mathbf{1}_{t \geq \tau} - \lambda t$$

This last class of models generates exogeneous default events independent of the stock price level.

We believe the CEV model and its extensions could be useful for pricing and understanding the growing equity credit-related market.

Appendix : CEV and Bessel Processes

The law of a CEV diffusion can be thought of in terms of squared Bessel process in the following way for: $t < \tau$,

$$S_t = e^{rt} R^{\frac{1}{2(1-\alpha)}}_{\left(\frac{2\alpha-1}{\alpha-1}, x_0^{2(1-\alpha)}\right)} \left(\frac{(1-\alpha)\sigma^2}{2r} (1 - e^{-2(1-\alpha)rt}) \right)$$

where $R_{(\delta,x)}$ is a squared Bessel Process of dimension δ and starting from x solution of

$$R_{(\delta,x)}(t) = x + \delta t + 2 \int_0^t \sqrt{R_{(\delta,x)}(u)} dW_u$$

where W is a brownian motion.
Next, we are interested in the law of the stopped CEV diffusion, thanks to Girsanov theorem, we obtain for a squared Bessel process R with $\mathcal{R}_t$ its canonical filtration

$$\mathbb{P}^{\delta}_{x|\mathcal{R}_t \cap \{t<\tau\}} = \left(\frac{R_{(4-\delta,x)}(t)}{x} \right)^{\frac{\delta}{2}-1} \cdot \mathbb{P}^{4-\delta}_{x|\mathcal{R}_t} \tag{3.11}$$

We can also get from Laplace transforms (see for example Delbaen and Shirakawa (2002)) the law of a squared Bessel process in terms of noncentral chi-square random variables:

$$R_{(\delta,x)}(t) \overset{(d)}{=} t V^{(\delta, \frac{x}{t})}$$

where $V^{(a,b)}$ is a noncentral chi-square r.v with a degrees of freedom and noncentrality parameter $b \geq 0$.

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Chapter 4

Options on Hedge Funds under the High-Water Mark Rule

[*Joint work with Hélyette Geman and Marc Yor; submitted for publication*]

The rapidly growing hedge fund industry has provided individual and institutional investors with new investment vehicles and styles of management. It has also brought forward a new form of performance contract: hedge fund managers receive incentive fees which are typically a fraction of the fund net asset value (NAV) above its starting level - a rule known as *high water mark*.

Options on hedge funds are becoming increasingly popular, in particular because they allow investors with limited capital to get exposure to this new asset class. The goal of the paper is to propose a valuation of plain-vanilla options on hedge funds which accounts for the high water market rule. Mathematically, this valuation leads to an interesting use of local times of Brownian motion. Option prices are numerically computed by inversion of their Laplace transforms.

4.1 Introduction

The term hedge fund is used to characterize a broad class of "skill-based" asset management firms that do not qualify as mutual funds regulated by the Investment Company Act of 1940 in the United States. Hedge funds are pooled investment vehicles that are privately organized, administered by professional investment managers and not widely available to the general public. Due to their private nature, they carry much fewer restrictions on the use of leverage, short-selling and derivatives than more regulated vehicles.

Across the nineties, hedge funds have been embraced by investors worldwide and are today recognized as an asset class in its own right. Originally, they were operated by taking a "hedged" position against a particular event, effectively reducing the overall risk. Today, the hedge component has totally disappeared and the term "hedge fund" refers to any pooled investment vehicle that is not a conventional fund using essentially long strategies in equity, bonds and money market instruments.

Over the recent years, multi-strategy funds of funds have in turn flourished, providing institutional investors with a whole spectrum of alternative investments exhibiting low correlations with traditional asset classes. In a parallel manner, options on hedge funds have been growing in numbers and types, offering individual investors the possibility of acquiring exposure to hedge funds through a relatively low amount of capital paid upfront at inception of the strategy.

Hedge funds constitute in fact a very heterogeneous group with strategies as diverse as convertible arbitrage, global macro or long short equity. In all cases however, common characteristics may be identified such as long-term commitment of investors, active management and broad discretion granted to the fund manager over the investment style and asset classes selected. Accordingly, incentive fees represent a significant percentage of the performance - typically ranging from 5% to 20%. This performance is most generally measured according to the high-water mark rule, i.e., using as a reference benchmark the Net Asset Value (NAV) of the fund at the time of purchase of the shares or options written on the hedge fund.

So far, the academic literature on hedge funds has focused on such issues as non-normality of returns, actual realized hedge fund performance and persistence of that performance. Amin and Kat (2003) show that, as a stand-alone investment, hedge funds do not offer a superior risk-return profile. Geman and Kharoubi (2003) propose instead the introduction of copulas to better represent the dependence structure between hedge funds and other asset classes. Agarwal and Naik (2000) examine whether persistence is sensitive to the length of the return measurement period and find maximum persistence at a quarterly horizon.

Another stream of papers has analyzed performance incentives in the hedge fund industry (see Fung and Hsieh (1999), Brown, Goetzmann and Ibbotson (1999)). However, the high water mark rule specification has been essentially studied by Goetzman, Ingersoll and Ross (2003).

High-water mark provisions condition the payment of the performance fee upon the hedge fund Net Asset Value exceeding the entry point of the investor. Goetzmann et al examine the costs and benefits to investors of this form of managers' compensation and the consequences of thess option-like characteristics on the values of fees on one hand, investors' claims on the other hand. Our objective is to pursue this analysis one step further and examine the valuation of options on hedge funds under the high-water mark rule. We show that this particular compound option-like problem may be solved in the Black-Scholes (1973) and Merton (1973) setting of geometric Brownian motion for the hedge fund NAV by the use of Local times of Brownian motion.

The remainder of the paper is organized as follows: Section II contains the description of the Net Asset Value dynamics, management and incentive fees and the NAV option valuation. Section II also extends the problem to a moving high water mark. Section III describes numerical examples obtained by inverse Laplace transforms and Monte Carlo simulations. Section IV contains concluding comments.

4.2 The High-Water Mark Rule and Local Times

4.2.1 Modeling the High-Water Mark

We work in a continuous-time framework and assume that the fund Net Asset Value (NAV) follows a lognormal diffusion process. This diffusion process will have a different starting point for each investor, depending on the time she entered her position. This starting point will define the high water mark used as the benchmark triggering the performance fees discussed throughout the paper.

We follow Goetzmann, Ingersoll and Ross (2003) in representing the performance fees in the following form

$$f(S_t) = \mu a \, \mathbf{1}_{\{S_t > H\}} \tag{4.1}$$

where S_t denotes the Net Asset Value at date t, μ is a mean NAV return statistically observed, a is a percentage generally comprised between 5% and 20% and $H = S_0$ denotes the market value of the NAV as observed at inception of the option contract.

We consider $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq 0\}, \mathbb{P}_0)$ a filtered probability space where $(B_t)_{t\geq 0}$ is an $\{\mathcal{F}_t, t \geq 0\}$ Brownian motion.

We now consider an equivalent measure $\mathbb{Q}$ under which the Net Asset Value dynamics $(S_t)_{t\geq 0}$ satisfy the stochastic differential equation:

$$\frac{dS_t}{S_t} = (r + \alpha - c - f(S_t))dt + \sigma dW_t \tag{4.2}$$

and the instantaneously compounding interest rate r is supposed to be constant. α denotes the excess return on the fund's assets and is classically defined by

$$\alpha = \mu - r - \beta (r_m - r)$$

where r_m is the expected return on the market portfolio. Hence, the "risk-neutral" return on the fund NAV is equal to $(r + \alpha)^*$; σ denotes the NAV volatility.

The management fees paid regardless of the performance are represented by a constant fraction c (comprised in practice between 0.5% and 2%) of the Net Asset Value. We represent the incentive fees as a deterministic function f of the current value S_t of the NAV, generally chosen according to the high water mark rule defined in equation (1). We can note that management fees have the form of the constant dividend payment of the Merton (1973) model while performance fees may be interpreted as a more involved form of dividend paid to the manager.

Because of their central role in what follows, we introduce the maximum and the minimum processes of the Brownian motion B, namely

$$M_t = \sup_{s\leq t} B_s, \; I_t = \inf_{s\leq t} B_s$$

as well as its local time at the level a, $a \in \mathbb{R}$

$$L_t^a = \lim_{\epsilon\to 0} \frac{1}{2\epsilon} \int_0^t \mathbf{1}_{\{|B_s - a| \leq \epsilon\}} ds$$

We also consider $A_t^{(a,+)} = \int_0^t \mathbf{1}_{\{B_s \geq a\}} ds$ and $A_t^{(a,-)} = \int_0^t \mathbf{1}_{\{B_s \leq a\}} ds$, respectively denoting the time spent in $[a; \infty[$ and the time spent in $]-\infty; a]$ by the Brownian motion up to time t.
For simplicity, we shall write $L_t = L_t^0$, $A_t^+ = A_t^{(0,+)}$ and $A_t^- = A_t^{(0,-)}$ the corresponding quantities for $a = 0$.

* Our claim is that the measure $\mathbb{Q}$ incorporates the price of market risk as a whole but not the excess performance - the fund "alpha" - achieved by the manager through the selection of specific securities at a given point in time. This view is in agreement with the footnote 6 in Goetzmann, Ingersoll and Ross (2003)

In order to extend our results to different types of incentive fees, we do not specify the function f but only assume that it is a bounded, increasing and positive function satisfying the following conditions:

$$f(0) = 0, \qquad \lim_{x\to\infty} f(x) < +\infty$$

Proposition 4.2.1 *There exists a unique solution to the stochastic differential equation*

$$\frac{dS_t}{S_t} = (r + \alpha - c - f(S_t))dt + \sigma dW_t$$

Proof. Let us denote $Y_t = \frac{\ln(S_t)}{\sigma}$. Applying Itô's formula, we see that the process Y_t satisfies the equation

$$dY_t = dW_t + \psi(e^{\sigma Y_t})dt$$

where $\psi(x) = r - \frac{\sigma^2}{2} + \alpha - c - f(x)$.
f, hence ψ is a Borel bounded function; consequently, we may apply Zvonkin (1974) theorem and obtain strong existence and pathwise uniqueness of the solution of equation (4.2).
We recall that Zvonkin theorem establishes that for every bounded Borel function ξ, the stochastic differential equation

$$dZ_t = dW_t + \xi(Z_t)dt$$

has a unique solution which is strong, i.e.: in this case, the filtration of Z and W are equal. ■
Integrating equation (4.2), we observe that this unique solution can be written as

$$S_t = S_0 \exp\left(\left(r + \alpha - c - \frac{\sigma^2}{2}\right) t - \int_0^t f(S_u)du + \sigma W_t\right)$$

We now seek to construct a new probability measure $\mathbb{P}$ under which the expression of S_t reduces to

$$S_t = S_0 \exp(\sigma \widetilde{W}_t) \tag{4.3}$$

where $\widetilde{W}_t$ is a $\mathbb{P}$ standard Brownian motion.

Proposition 4.2.2 *There exists an equivalent measure $\mathbb{P}$ under which the Net Asset Value dynamics satisfy the stochastic differential equation*

$$\frac{dS_t}{S_t} = \frac{\sigma^2}{2}dt + \sigma d\widetilde{W}_t \tag{4.4}$$

where

$$\mathbb{Q}_{|\mathcal{F}_t} = Z_t \cdot \mathbb{P}_{|\mathcal{F}_t} \tag{4.5}$$

$$Z_t = \exp\big(\int_0^t \big(b - \frac{f(S_u)}{\sigma}\big)d\widetilde{W}_u - \frac{1}{2}\int_0^t \big(b - \frac{f(S_u)}{\sigma}\big)^2 du\big)$$

and

$$b = \frac{r + \alpha - c - \frac{\sigma^2}{2}}{\sigma}$$

Proof. Thanks to Girsanov theorem (see for instance McKean (1969) and Revuz and Yor (2005)) we find that under the probability measure $\mathbb{P}$,
$\widetilde{W}_t = W_t + \int_0^t du\big(b - \frac{f(e^{\sigma Y_u})}{\sigma}\big)$ is a Brownian motion, which allows us to conclude. ■

4.2.2 Building the Pricing Framework

For practical purposes, the issuer of the call is typically the hedge fund itself, hence hedging arguments allow to price the option as the expectation (under the right probability measure) of the discounted payoff. More generally, a European-style hedge fund derivative with maturity $T > 0$ is defined by its payoff $F : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ and the valuation of the option reduces to computing expectations of the following form:

$$V_F(t, S, T) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\big[F(S_u; u \leq T)\big|\mathcal{F}_t\big]$$

For the case where the valuation of the option takes place at a date $t = 0$, we denote $V_F(S, T) = V_F(0, S, T)$. We can observe that we are in a situation of complete markets since the only source of randomness is the Brownian motion driving the NAV dynamics.

Proposition 4.2.3 *For any payoff F that can be written as an increasing function of the stock price process, the option price associated to the above payoff is an increasing function of the high-water mark level.*

Proof. This result is quite satisfactory from a financial perspective. Mathematically, it may be deduced from the following result :
Let us consider the solutions (S^1, S^2) of the pair of stochastic differential equations :

$$\begin{aligned} dS^1_t &= b^1(S^1_t)dt + \sigma S^1_t dW_t \\ dS^2_t &= b^2(S^2_t)dt + \sigma S^2_t dW_t \end{aligned}$$

where

$$\begin{aligned} b^1(x) &= (r + \alpha - c - \mu a\ \mathbf{1}_{\{x>H\}})x \\ b^2(x) &= (r + \alpha - c - \mu a\ \mathbf{1}_{\{x>H'\}})x \end{aligned}$$

with $H > H'$ and $S^1_0 = S^2_0$ a.s.
We may apply a comparison theorem since b^1 and b^2 are bounded Borel functions and $b^1 \geq b^2$ everywhere, obtain that

$$\mathbb{P}[S^1_t \geq S^2_t; \forall t \geq 0] = 1$$

and then conclude. ■

If we consider a call option and a put option with strike K and maturity T, we observe the following call-put parity relation:

$$C_0(K, T) - P_0(K, T) = \mathbb{E}^{\mathbb{Q}}[e^{-rT}S_T] - Ke^{-rT} \tag{4.6}$$

We now wish to express the exponential $(\mathcal{F}_t, \mathbb{P})$-martingale Z_t featured in (4.5) in terms of well-known processes in order to be able to obtain closed-form pricing formulas.

Lemma 4.2.4 *Let us define d_H, λ, α_+, α_- and ϕ as follows:*

$$d_H = \frac{\ln(\frac{H}{S_0})}{\sigma}, \qquad \lambda = \frac{\mu a}{2\sigma}$$

$$\alpha_+ = 2\lambda^2 + \frac{b^2}{2} - 2\lambda b, \qquad \alpha_- = \frac{b^2}{2}$$
$$\phi(x) = e^{bx - 2\lambda(x-d_H)_+}$$

We then obtain:

$$Z_t = e^{2\lambda(-d_H)_+}\phi(\widetilde{W}_t)\exp(\lambda L_t^{d_H})\exp(-\alpha_+ A_t^{(d_H,+)} - \alpha_- A_t^{(d_H,-)}) \tag{4.7}$$

Proof. The proof of this proposition is based on the one hand on the Tanaka formula which, for a Brownian motion B and any real number a, establishes that

$$(B_t - a)_+ = (-a)_+ + \int_0^t dB_s \mathbf{1}_{\{B_s > a\}} + \frac{1}{2} L_t^a$$

On the other hand, we can rewrite

$$f(S_t) = \mu a \mathbf{1}_{\{\widetilde{W}_t > d_H\}}$$

Observing that $A_t^{(d_H,+)} + A_t^{(d_H,-)} = t$ leads to the result. ■
From the above lemma, we obtain that:

$$\begin{aligned} V_F(S,T) &= e^{-rT}\mathbb{E}^{\mathbb{P}}\big[Z_T F(S_u; u \le T)\big] \\ &= e^{-rT+2\lambda(-d_H)_+}\mathbb{E}^{\mathbb{P}}\big[\phi(W_T)\exp(\lambda L_T^{d_H} - \alpha_+ A_T^{(d_H,+)} - \alpha_- A_T^{(d_H,-)})F(S_0 e^{\sigma \widetilde{W}_u}; u \le T)\big] \end{aligned}$$

The price of a NAV call option is closely related to the law of the triple $(W_t, L_t^a, A_t^{(a,+)})$. Karatzas and Shreve (1991) have extensively studied this joint density for $a = 0$ and obtained in particular the following remarkable result

Proposition 4.2.5 *For any positive t and b, $0 < \tau < t$, we have*

$$\begin{aligned} \mathbb{P}[W_t \in dx; L_t \in db, A_t^+ \in d\tau] &= f(x, b; t, \tau)\ dx\ db\ d\tau; \qquad x > 0 \\ &= f(-x, b; t, -\tau)\ dx\ db\ d\tau; \qquad x < 0 \end{aligned}$$

where

$$f(x, b; t, \tau) = \frac{b(2x+b)}{8\pi\tau^{\frac{3}{2}}(t-\tau)^{\frac{3}{2}}}\exp\left(-\frac{b^2}{8(t-\tau)} - \frac{(2x+b)^2}{8\tau}\right)$$

This formula could lead to a computation of the option price based on a multiple integration but it would be numerically intensive; moreover, obtaining an analytical formula for the triple integral involved in the option price seems quite unlikely. We observe instead that in the above density f, a convolution product appears, which leads us to compute either Fourier or Laplace transforms. We are in fact going to compute the Laplace transform with respect to time to maturity of the option price. This way to proceed is mathematically related to the Karatzas and Shreve result in Proposition 4.2.5. In the same way, we can notice that the Laplace transform exhibited by Geman and Yor (1996) for the valuation of a Double Barrier option is related to the distribution of the triple (W_t, M_t, I_t) Brownian motion, its maximum and minimum used by Kunitomo and Ikeda (1992) for the same pricing problem. The formulas involved in the NAV call price rely on the following result which may be obtained from Brownian excursion theory:

Proposition 4.2.6 *Let W_t be a standard Brownian motion, L_t its local time at zero, A_t^+ and A_t^- the times spent positively and negatively until time t. For any function $h \in L^1(\mathbb{R})$, the Laplace transform of the quantity $g(t) = \mathbb{E}\big[h(W_t)\exp(\lambda L_t)\exp(-\mu A_t^+ - \nu A_t^-)\big]$ has the following analytical expression*

$$\int_0^\infty dte^{-\frac{\theta}{2}t} g(t) = 2\frac{\left(\int_0^\infty dxe^{-x\sqrt{\theta+2\mu}}h(x) + \int_0^\infty dxe^{-x\sqrt{\theta+2\nu}}h(-x)\right)}{\sqrt{\theta+2\mu}+\sqrt{\theta+2\nu}-2\lambda}$$

for θ large enough to ensure positivity of the denominator.

Proof. See the Appendix for details. The result is rooted in the theory of excursions of the Brownian motion. ■

4.2.3 Valuation of the Option at Inception of the Contract

In this section, we turn to the computation of the price of a European call option written on a Hedge Fund NAV under the high-water mark rule. Consequently, the payoff considered is the following:

$$F(S_u; u \le T) = (S_T - K)_+ \tag{4.8}$$

or, in a more convenient way for our purpose

$$F(\widetilde{W}_u; u \le T) = (S_0 \exp(\sigma \widetilde{W}_T) - K)_+$$

At inception of the contract, the high-water mark that is chosen is the spot price, hence $H = S_0$ and $d_H = 0$. This specific framework allows us to use fundamental results on the joint law of the triple (B_t, L_t^0, A_t^+) presented in Proposition 4.2.6. We write the European call option price as follows

$$C(0, S_0) = e^{-rT}\mathbb{E}^{\mathbb{P}}\big[h(\widetilde{W}_T)\exp(\lambda L_T - \alpha_+ A_T^+ - \alpha_- A_T^-)\big]$$

where $h(x) = (S_0 e^{\sigma x} - K)_+ e^{bx - 2\lambda(x)_+}$.

We now compute the Laplace transform in time to maturity of the European call option on the NAV of an Hedge Fund, that is to say the following quantity:

$$\begin{aligned} \forall \theta \in \mathbb{R}_+ \qquad I(\theta) &= \int_0^\infty dte^{-\frac{\theta}{2}t} e^{-rt}\mathbb{E}^{\mathbb{Q}}[(S_t - K)_+] \\ &= \int_0^\infty dte^{-(\frac{\theta}{2}+r)t}\mathbb{E}^{\mathbb{P}}[Z_t(S_t - K)_+] \end{aligned}$$

Lemma 4.2.7 *The Laplace transform with respect to time to maturity of a call option price has the following analytical expression:*

$$I(\theta) = 2\frac{\left(\int_0^\infty dxe^{-x\sqrt{\theta+2(r+\alpha_+)}}h(x) + \int_0^\infty dxe^{-x\sqrt{\theta+2(r+\alpha_-)}}h(-x)\right)}{\sqrt{\theta+2(r+\alpha_+)}+\sqrt{\theta+2(r+\alpha_-)}-2\lambda} \tag{4.9}$$

where $h(x) = e^{bx - 2\lambda x_+}(S_0 e^{\sigma x} - K)_+$

Proof. We obtain from Lemma 4.2.4 that:

$$\mathbb{E}^{\mathbb{P}}[Z_t(S_t - K)_+] = \mathbb{E}\big[h(\widetilde{W}_t)\exp(\lambda L_t)\exp(-\alpha_+ A_t^+ - \alpha_- A_t^-)\big]$$

where:

$$h(x) = e^{bx - 2\lambda x_+}(S_0 e^{\sigma x} - K)_+$$

Then, using Proposition 4.2.6, we are able to conclude. ∎

This lemma leads us to compute explicit formulas for the Laplace transform of a call option that is in-the-money ($S_0 \geq K$) at date 0 and out-of-the-money ($S_0 < K$) that we present in two consecutive propositions.

Proposition 4.2.8 *For an out-of-the-money call option ($S_0 \leq K$), the Laplace transform of the price is given by the following formula:*

$$I(\theta) = \frac{N(\theta)}{D(\theta)}$$

where

$$\theta > (\sigma + b - 2\lambda)^2 - 2(r + \alpha_+)$$

and

$$\begin{aligned} D(\theta) &= \frac{\sqrt{\theta + 2(r+\alpha_+)} + \sqrt{\theta + 2(r+\alpha_-)} - 2\lambda}{2} \\ N(\theta) &= \frac{S_0}{\sqrt{\theta + 2(r+\alpha_+)} + 2\lambda - \sigma - b}\left(\frac{S_0}{K}\right)^{\frac{\sqrt{\theta+2(r+\alpha_+)}+2\lambda-\sigma-b}{\sigma}} \\ &\quad - \frac{K}{\sqrt{\theta + 2(r+\alpha_+)} + 2\lambda - b}\left(\frac{S_0}{K}\right)^{\frac{\sqrt{\theta+2(r+\alpha_+)}+2\lambda-b}{\sigma}} \end{aligned}$$

Proof. Keeping the notation of Proposition 4.2.6, we can write

$$\forall x > 0, \qquad h(x) = (S_0 e^{\sigma x} - K)\mathbf{1}_{\{x \geq \frac{1}{\sigma}\ln(\frac{K}{S_0})\}} e^{(b-2\lambda)x} \qquad \textit{and} \qquad h(-x) = 0$$

and then by simple integration, obtain the stated formula. ∎

Proposition 4.2.9 *For an in-the-money call option ($S_0 \geq K$), the Laplace transform of the price is given by the following formula:*

$$I(\theta) = \frac{N_1(\theta) + N_2(\theta)}{D(\theta)}$$

where $\theta > (\sigma + b - 2\lambda)^2 - 2(r + \alpha_+)$ *and*

$$\begin{aligned}
D(\theta) &= \frac{\sqrt{\theta + 2(r+\alpha_+)} + \sqrt{\theta + 2(r+\alpha_-)} - 2\lambda}{2} \\
N_1(\theta) &= \frac{S_0}{\sqrt{\theta + 2(r+\alpha_+)} + 2\lambda - \sigma - b} - \frac{K}{\sqrt{\theta + 2(r+\alpha_+)} + 2\lambda - b} \\
N_2(\theta) &= \frac{S_0}{\sqrt{\theta + 2(r+\alpha_-)} + \sigma + b}\left(1 - \left(\frac{K}{S_0}\right)^{\frac{\sqrt{\theta+2(r+\alpha_-)}+\sigma+b}{\sigma}}\right) \\
&\quad - \frac{K}{\sqrt{\theta + 2(r+\alpha_-)} + b}\left(1 - \left(\frac{K}{S_0}\right)^{\frac{\sqrt{\theta+2(r+\alpha_-)}+b}{\sigma}}\right),
\end{aligned}$$

α_- *and* α_+ *being defined in Lemma 4.2.4 .*

Proof. We have

$$\forall x > 0, \qquad h(x) = (S_0 e^{\sigma x} - K)e^{(b-2\lambda)x} \qquad \textit{and} \qquad h(-x) = (S_0 e^{-\sigma x} - K)\mathbf{1}_{\{x \le \frac{1}{\sigma}\ln(\frac{S_0}{K})\}} e^{-bx}$$

and as in the previous proposition, the Laplace transform is derived. ■

As a side note, we observe that the case $K = 0$ provides the Laplace transform of the $t-$maturity forward contract written on the NAV at date 0

$$\int_0^\infty dt e^{-\frac{\theta}{2}t}\mathbb{E}^{\mathbb{P}}[e^{-rt}S_t] = 2\frac{\frac{S_0}{\sqrt{\theta+2(r+\alpha_+)}+2\lambda-\sigma-b} + \frac{S_0}{\sqrt{\theta+2(r+\alpha_-)}+\sigma+b}}{\sqrt{\theta + 2(r+\alpha_+)} + \sqrt{\theta + 2(r+\alpha_-)} - 2\lambda}$$

where $\theta > (\sigma + b - 2\lambda)^2 - 2(r + \alpha_+)$.

It is satisfactory to check that by choosing $a = 0$, $\alpha = 0$, we obtain the Laplace transform of a European call option on a dividend-paying stock with a continuous dividend yield c whose dynamics satisfy as in Merton (1973), the equation

$$\frac{dS_t}{S_t} = (r - c)\ dt + \sigma\ dW_t$$

This Laplace transform is derived from Proposition 4.2.8 for an out-of-the-money call option and from Proposition 4.2.9 for an in-the-money call option.

4.2.4 Valuation during the lifetime of the Option

Evaluating at a time t a call option on a hedge fund written at date 0 implies that we are in the situation where $d_H = \frac{1}{\sigma}\ln(\frac{H}{S_t})$ may be different from 0. Since the solution of the stochastic differential equation driving the Net Asset Value is a Markov process, the evaluation of the option at time t only depends on the value of the process at time t and on the time to maturity $T - t$. Hence, we need to compute the following quantity

$$C(t, S_t) = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}(S_T - K)_+ | \mathcal{F}_t\right]$$

Given the relationship between $\mathbb{P}$ and $\mathbb{Q}$, we can write

$$C(t,S_t) = e^{-r(T-t)}e^{2\lambda(-d_H)_+}\mathbb{E}^{\mathbb{P}}\big[h(\widetilde{W}_{T-t})\exp(\lambda L^{d_H}_{T-t} - \alpha_+ A^{(d_H,+)}_{T-t} - \alpha_- A^{(d_H,-)}_{T-t})\big]$$

where $h(x) = e^{bx-2\lambda(x-d_H)_+}(S_t e^{\sigma x} - K)_+$

Because of the importance of the level d_H in the computations, we introduce the stopping time $\tau_{d_H} = \inf\{t \geq 0; \widetilde{W}_t = d_H\}$ and split the problem into the computation of the two following quantities:

$$\begin{aligned} C_1 &= e^{-r(T-t)}e^{2\lambda(-d_H)_+}\mathbb{E}^{\mathbb{P}}\big[\mathbf{1}_{\{\tau_{d_H}>T-t\}}h(\widetilde{W}_{T-t})\exp(\lambda L^{d_H}_{T-t} - \alpha_+ A^{(d_H,+)}_{T-t} - \alpha_- A^{(d_H,-)}_{T-t})\big] \\ &\text{and} \\ C_2 &= e^{-r(T-t)}e^{2\lambda(-d_H)_+}\mathbb{E}^{\mathbb{P}}\big[\mathbf{1}_{\{\tau_{d_H}<T-t\}}h(\widetilde{W}_{T-t})\exp(\lambda L^{d_H}_{T-t} - \alpha_+ A^{(d_H,+)}_{T-t} - \alpha_- A^{(d_H,-)}_{T-t})\big] \end{aligned}$$

In order to compute C_1, we introduce for simplicity $s = T-t$ and obtain

$$\begin{aligned} \mathbb{E}^{\mathbb{P}}\big[\mathbf{1}_{\{\tau_{d_H}>s\}}h(\widetilde{W}_s)e^{\lambda L^{d_H}_s - \alpha_+ A^{(d_H,+)}_s - \alpha_- A^{(d_H,-)}_s}\big] &= e^{-s\alpha_-}\mathbb{E}^{\mathbb{P}}\big[\mathbf{1}_{\{M_s<d_H\}}h(\widetilde{W}_s)\big] \quad \text{if} \quad d_H > 0 \\ &= e^{-s\alpha_+}\mathbb{E}^{\mathbb{P}}\big[\mathbf{1}_{\{I_s>d_H\}}h(\widetilde{W}_s)\big] \quad \text{if} \quad d_H < 0 \end{aligned}$$

We now need to recall some well-known results on Brownian motion first-passage times that one may find for instance in Karatzas and Shreve (1991).

Lemma 4.2.10 *The following equalities hold for $u > 0$ and $a > 0$*

$$\mathbb{P}[\tau_a \leq u] = \mathbb{P}[M_u \geq a] = \frac{2}{\sqrt{2\pi}}\int_{\frac{a}{\sqrt{u}}}^{\infty} e^{-\frac{x^2}{2}}\,dx$$

Hence, for $u > 0$ and $a \in \mathbb{R}$

$$\mathbb{P}[\tau_a \in du] = \frac{|a|}{\sqrt{2\pi u^3}}e^{-\frac{a^2}{2u}}\,du$$

and for $\lambda > 0$

$$\mathbb{E}[e^{-\lambda\tau_a}] = e^{-|a|\sqrt{2\lambda}}$$

where $\tau_a = \inf\{t \geq 0; W_t = a\}$

Lemma 4.2.11 *For $b \geq 0$ and $a \leq b$, the joint density of (W_u, M_u) is given by :*

$$\mathbb{P}[W_u \in da, M_u \in db] = \frac{2(2b-a)}{\sqrt{2\pi u^3}}\exp\Big\{-\frac{(2b-a)^2}{2u}\Big\}da\;db$$

and likewise, for $b \leq 0$ and $a \geq b$ the joint density of (W_u, I_u) is given by

$$\mathbb{P}[W_u \in da, I_u \in db] = \frac{2(a-2b)}{\sqrt{2\pi u^3}}\exp\Big\{-\frac{(2b-a)^2}{2u}\Big\}dadb$$

These lemmas provide us with the following interesting property

Proposition 4.2.12 *Let us consider W_u a standard Brownian motion, I_u and M_u respectively its minimum and maximum values up to time u.*
For any function $h \in L^1(\mathbb{R})$, the quantity $k_a(u) = \mathbb{E}\big[\mathbf{1}_{\{\tau_a > u\}} h(W_u)\big]$ is given by

$$\int_{-\infty}^{\frac{a}{\sqrt{u}}} dv \frac{e^{-\frac{v^2}{2}}}{\sqrt{2\pi}} h(v\sqrt{u}) - \int_{-\infty}^{-\frac{a}{\sqrt{u}}} dv \frac{e^{-\frac{v^2}{2}}}{\sqrt{2\pi}} h(v\sqrt{u} + 2a) \quad \text{if} \quad a > 0$$

$$\int_{-\infty}^{-\frac{a}{\sqrt{u}}} dv \frac{e^{-\frac{v^2}{2}}}{\sqrt{2\pi}} h(-v\sqrt{u}) - \int_{-\infty}^{\frac{a}{\sqrt{u}}} dv \frac{e^{-\frac{v^2}{2}}}{\sqrt{2\pi}} h(-v\sqrt{u} + 2a) \quad \text{if} \quad a < 0$$

Proof. We first observe that

$$\begin{aligned} \mathbb{E}^{\mathbb{P}}\big[\mathbf{1}_{\{\tau_a > u\}} h(W_u)\big] &= \mathbb{E}^{\mathbb{P}}\big[\mathbf{1}_{\{M_u < a\}} h(W_u)\big] \quad \text{if} \quad a > 0 \\ &= \mathbb{E}^{\mathbb{P}}\big[\mathbf{1}_{\{I_u > a\}} h(W_u)\big] \quad \text{if} \quad a < 0 \end{aligned}$$

By symmetry, we only need to show the result in the case $a > 0$. From the previous lemma, we can write

$$\mathbb{E}\big[\mathbf{1}_{\{M_u < a\}} h(W_u)\big] = \int_0^a db \int_{-\infty}^b dx h(x) \frac{2(2b - x)}{\sqrt{2\pi u^3}} \exp\big\{ - \frac{(2b - x)^2}{2u} \big\}$$

Finally, we conclude by applying Fubini's theorem. ■
As a consequence, we can now compute the quantity C_1

Proposition 4.2.13 *For a call option such that $d_H > 0$ or equivalently $H > S_t$, the quantity C_1 is equal to*

$$e^{-(r+\alpha_-)s} G(K, H, S_t, s)$$

where $s = T - t$ and

$$\begin{aligned} G(K, H, S_t, s) &= 0 \quad if \quad K \geq H \\ G(K, H, S_t, s) &= S_t e^{s\frac{(b+\sigma)^2}{2}} N_1 - K e^{s\frac{b^2}{2}} N_2 \quad if \quad K < H \\ N_1 &= N\Big(\frac{d_H}{\sqrt{s}} - \sqrt{s}(b + \sigma)\Big) - N\Big(\frac{d_K}{\sqrt{s}} - \sqrt{s}(b + \sigma)\Big) \\ &\quad - e^{2(b+\sigma)d_H}\Big(N\Big(-\frac{d_H}{\sqrt{s}} - \sqrt{s}(b + \sigma)\Big) - N\Big(\frac{d_K - 2d_H}{\sqrt{s}} - \sqrt{s}(b + \sigma)\Big)\Big) \\ N_2 &= N\Big(\frac{d_H}{\sqrt{s}} - \sqrt{s}b\Big) - N\Big(\frac{d_K}{\sqrt{s}} - \sqrt{s}b\Big) \\ &\quad - e^{2bd_H}\Big(N\Big(-\frac{d_H}{\sqrt{s}} - \sqrt{s}b\Big) - N\Big(\frac{d_K - 2d_H}{\sqrt{s}} - \sqrt{s}b\Big)\Big) \end{aligned}$$

where $N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x dy e^{-\frac{y^2}{2}}$

Proof. We apply Proposition 4.2.12 in the case $a > 0$ with $h(x) = (S_0 e^{\sigma x} - K)_+ e^{bx - 2\lambda(x - d_H)_+}$.
■

Proposition 4.2.14 *For a call option such that $d_H < 0$ or equivalently $H < S_t$, the quantity C_1 is given by*

$$e^{-(r+\alpha_+)s} J(K, H, S_t, s)$$

where $s = T - t$

$$\begin{aligned} J(K,H,S_t,s) &= S_t e^{s\frac{(b-2\lambda+\sigma)^2}{2}} N_1(d_1,d_2) - K e^{s\frac{(b-2\lambda)^2}{2}} N_2(d_1,d_2) \\ N_1(d_1,d_2) &= N(-\frac{d_1}{\sqrt{s}} + \sqrt{s}(b+\sigma-2\lambda)) - e^{2(b+\sigma-2\lambda)d_H} N(\frac{d_2}{\sqrt{s}} + \sqrt{s}(b+\sigma-2\lambda)) \\ N_2(d_1,d_2) &= N(-\frac{d_1}{\sqrt{s}} + \sqrt{s}(b-2\lambda)) - e^{2(b-2\lambda)d_H} N(\frac{d_2}{\sqrt{s}} + \sqrt{s}(b-2\lambda)) \\ (d_1,d_2) &= (d_K, 2d_H - d_K) \quad if \quad K > H \\ (d_1,d_2) &= (d_H, d_H) \quad if \quad K \le H \end{aligned}$$

where $N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} dy e^{-\frac{y^2}{2}}$

Proof. We apply Proposition 4.2.12 in the case $a < 0$ with $h(x) = (S_0 e^{\sigma x} - K)_+ e^{bx - 2\lambda(x-d_H)_+}$. ■

In order to compute C_2, it is useful to exhibit a result similar to the one obtained in Proposition 4.2.5 to obtain the Laplace transform of the joint density of $(B_t, L_t^a, A_t^{(a,+)}, A_t^{(a,-)})$.

Proposition 4.2.15 *Let us consider W_t a standard Brownian motion, L_t^a its local time at the level a, $A_t^{(a,+)}$ and $A_t^{(a,-)}$ respectively the time spent above and below a by the Brownian motion W until time t.*
For any function $h \in L^1(\mathbb{R})$, the Laplace Transform $\int_0^\infty dt e^{-\frac{\theta}{2}t} g_a(t)$ of the quantity $g_a(t) = \mathbb{E}\big[\mathbf{1}_{\{\tau_a < t\}} h(W_t) \exp(\lambda L_t^a) \exp(-\mu A_t^{(a,+)} - \nu A_t^{(a,-)})\big]$ is given by

$$\begin{aligned} &2e^{-a\sqrt{\theta+2\nu}} \frac{\left(\int_0^\infty dx e^{-x\sqrt{\theta+2\mu}} h(a+x) + \int_0^\infty dx e^{-x\sqrt{\theta+2\nu}} h(a-x)\right)}{\sqrt{\theta+2\mu} + \sqrt{\theta+2\nu} - 2\lambda} \quad if \quad a > 0 \\ &2e^{a\sqrt{\theta+2\mu}} \frac{\left(\int_0^\infty dx e^{-x\sqrt{\theta+2\mu}} h(a+x) + \int_0^\infty dx e^{-x\sqrt{\theta+2\nu}} h(a-x)\right)}{\sqrt{\theta+2\mu} + \sqrt{\theta+2\nu} - 2\lambda} \quad if \quad a < 0 \end{aligned}$$

for θ large enough, as seen before.

Proof. Let us prove this result in the case $a > 0$; it easily yields to the case $a < 0$.
We first write

$$g_a(t) = e^{-\nu t} \mathbb{E}\big[\mathbf{1}_{\{\tau_a < t\}} h(W_t) \exp(\lambda L_t^a) e^{-(\mu-\nu) A_t^{(a,+)}}\big]$$

Then

$$I(\theta) = \int_0^{+\infty} dt\ e^{-t\frac{\theta+2\nu}{2}} \mathbb{E}\big[\mathbf{1}_{\{\tau_a < t\}} h(W_t) \exp(\lambda L_t^a) e^{-(\mu-\nu) A_t^{(a,+)}}\big]$$

We now use the strong Markov property and observe that $B_t = W_{t+\tau_a} - W_{\tau_a} = W_{t+\tau_a} - a$ is a Brownian motion. Next, we compute the quantity

$$\begin{aligned}
\mathbb{E}\big[\mathbf{1}_{\{\tau_a<t\}} h(W_t)\exp(\lambda L_t^a) e^{-(\mu-\nu)A_t^{(a,+)}}\big] &= \mathbb{E}\big[\mathbf{1}_{\{\tau_a<t\}} h(B_{t-\tau_a}+a)\exp(\lambda L_{t-\tau_a}) e^{-(\mu-\nu)A^+_{t-\tau_a}}\big] \\
&= \int_0^t ds \frac{ae^{-\frac{a^2}{2s}}}{\sqrt{2\pi s^3}} \mathbb{E}\big[h(B_{t-s}+a)\exp(\lambda L_{t-s}) e^{-(\mu-\nu)A^+_{t-s}}\big] \\
&= \int_0^t ds \frac{ae^{-\frac{a^2}{2(t-s)}}}{\sqrt{2\pi (t-s)^3}} \mathbb{E}\big[h(B_s+a)\exp(\lambda L_s) e^{-(\mu-\nu)A^+_s}\big]
\end{aligned}$$

since

$$L_t^a(a + B_{(\cdot-\tau_a)_+}) = L_{(t-\tau_a)_+}$$

$$A_t^{(a,+)} = \int_0^t ds \mathbf{1}_{\{B_{(s-\tau_a)_+}>0\}} = A^+_{(t-\tau_a)_+}$$

Hence, applying Fubini's theorem and Proposition 4.2.6 we obtain

$$\begin{aligned}
I(\theta) &= \int_0^\infty dse^{-\frac{\theta}{2}s}\mathbb{E}\big[h(a+B_s)\exp(\lambda L_s)\exp(-\mu A_s^+ - \nu A_s^-)\big] \int_0^\infty due^{-\frac{\theta+2\nu}{2}u}\frac{|a|e^{-\frac{a^2}{2u}}}{\sqrt{2\pi u^3}} \\
&= 2e^{-a\sqrt{\theta+2\nu}} \frac{\Big(\int_0^\infty dxe^{-x\sqrt{\theta+2\mu}}h(a+x) + \int_0^\infty dxe^{-x\sqrt{\theta+2\nu}}h(a-x)\Big)}{\sqrt{\theta+2\mu}+\sqrt{\theta+2\nu}-2\lambda}
\end{aligned}$$

■

Proposition 4.2.16 *In the case $H \le K$, the Laplace transform with respect to the variable $T-t$ of the quantity C_2 is given by the following formula:*

$$I(\theta) = M(\theta)\frac{N(\theta)}{D(\theta)}$$

where

$$\theta > (\sigma + b - 2\lambda)^2 - 2(r+\alpha_+)$$

and

$$\begin{aligned}
M(\theta) &= \left(\frac{H}{S_t}\right)^{\frac{b-\sqrt{\theta+2(r+\alpha_-)}}{\sigma}} \qquad if \quad H > S_t \\
M(\theta) &= \left(\frac{S_t}{H}\right)^{\frac{2\lambda-b-\sqrt{\theta+2(r+\alpha_+)}}{\sigma}} \qquad if \quad H < S_t \\
D(\theta) &= \frac{\sqrt{\theta+2(r+\alpha_+)}+\sqrt{\theta+2(r+\alpha_-)}-2\lambda}{2} \\
N(\theta) &= \frac{H}{\sqrt{\theta+2(r+\alpha_+)}+2\lambda-\sigma-b}\left(\frac{H}{K}\right)^{\frac{\sqrt{\theta+2(r+\alpha_+)}+2\lambda-\sigma-b}{\sigma}} \\
&\quad - \frac{K}{\sqrt{\theta+2(r+\alpha_+)}+2\lambda-b}\left(\frac{H}{K}\right)^{\frac{\sqrt{\theta+2(r+\alpha_+)}+2\lambda-b}{\sigma}}
\end{aligned}$$

Proof. We prove this result by applying Proposition 4.2.8 and Proposition 4.2.15 and noticing that $(S_0 e^{\sigma(x+d_H)} - K)_+ = (He^{\sigma x} - K)_+$ ■

Proposition 4.2.17 *In the case $H \geq K$, the Laplace transform with respect to the variable $T-t$ of the quantity C_2 is given by the formula:*

$$\begin{aligned}
I(\theta) &= M(\theta)\frac{N_1(\theta)+N_2(\theta)}{D(\theta)} \\
M(\theta) &= \left(\frac{H}{S_t}\right)^{\frac{b-\sqrt{\theta+2(r+\alpha_-)}}{\sigma}} \qquad if \quad H > S_t \\
M(\theta) &= \left(\frac{S_t}{H}\right)^{\frac{2\lambda-b-\sqrt{\theta+2(r+\alpha_+)}}{\sigma}} \qquad if \quad H < S_t \\
D(\theta) &= \frac{\sqrt{\theta+2(r+\alpha_+)}+\sqrt{\theta+2(r+\alpha_-)}-2\lambda}{2} \\
N_1(\theta) &= \frac{H}{\sqrt{\theta+2(r+\alpha_+)}+2\lambda-\sigma-b} - \frac{K}{\sqrt{\theta+2(r+\alpha_+)}+2\lambda-b} \\
N_2(\theta) &= \frac{H}{\sqrt{\theta+2(r+\alpha_-)}+\sigma+b}\left(1-(\frac{K}{H})^{\frac{\sqrt{\theta+2(r+\alpha_-)}+\sigma+b}{\sigma}}\right) \\
&\quad -\frac{K}{\sqrt{\theta+2(r+\alpha_-)}+b}\left(1-(\frac{K}{H})^{\frac{\sqrt{\theta+2(r+\alpha_-)}+b}{\sigma}}\right)
\end{aligned}$$

where $\theta > (\sigma+b-2\lambda)^2 - 2(r+\alpha_+)$

Proof. This result is immediately derived from Proposition 4.2.9 and Proposition 4.2.15. ■

4.2.5 Extension to a Moving High-Water Mark

We now wish to take into account the fact that the threshold triggering the performance fees may accrue at the risk-free rate. As a consequence, we define $\widetilde{f}$ as

$$\widetilde{f}(t, S_t) = \mu a \mathbf{1}_{\{S_t > He^{rt}\}}$$

Proposition 4.2.18 *There exists a unique solution to the stochastic differential equation*

$$\frac{dS_t}{S_t} = (r + \alpha - c - \widetilde{f}(t, S_t))dt + \sigma dW_t \tag{4.10}$$

Proof. Let us denote $Y_t = \frac{\ln(S_t e^{-rt})}{\sigma}$. Applying Itô's formula, we can see that Y_t satisfies the following equation

$$dY_t = dW_t + \psi(e^{\sigma Y_t})dt$$

where $\psi(x) = -\frac{\sigma^2}{2} + \alpha - c - f(x)$ and f denotes the performance fees function defined in equation (4.1).
ψ is Borel locally bounded, consequently we may again apply Zvonkin theorem that ensures strong existence and pathwise uniqueness of the solution of (4.10). ■

Let us denote $\widetilde{S}_t = S_t e^{-rt}$; we seek to construct a probability measure $\widehat{\mathbb{Q}}$ under which

$$\widetilde{S}_t = S_0 \exp(\sigma \widehat{W}_t)$$

where $\widehat{W}_t$ is a $\widehat{\mathbb{Q}}$ standard Brownian motion. We briefly extend the results of the previous section to the case of a moving high-water mark.

Proposition 4.2.19 *There exists an equivalent measure $\widehat{\mathbb{Q}}$ under which the Net Asset Value dynamics satisfy the stochastic differential equation*

$$\frac{dS_t}{S_t} = (r + \frac{\sigma^2}{2})dt + \sigma d\widehat{W}_t \tag{4.11}$$

Moreover,

$$\mathbb{Q}_{|\mathcal{F}_t} = Z_t \cdot \widehat{\mathbb{Q}}_{|\mathcal{F}_t} \tag{4.12}$$

where

$$Z_t = \exp\Big(\int_0^t \big(b - \frac{f(\widetilde{S}_u)}{\sigma}\big) d\widehat{W}_u - \frac{1}{2}\int_0^t \big(b - \frac{f(\widetilde{S}_u)}{\sigma}\big)^2 du\Big)$$

and

$$b = \frac{\alpha - c - \frac{\sigma^2}{2}}{\sigma}$$

Lemma 4.2.20 *Let us define d_H, λ, α_+, α_- and ϕ as follows:*

$$d_H = \frac{\ln(\frac{H}{S_0})}{\sigma}, \qquad \lambda = \frac{\mu a}{2\sigma}$$

$$\alpha_+ = 2\lambda^2 + \frac{b^2}{2} - 2\lambda b, \qquad \alpha_- = \frac{b^2}{2}$$

$$\phi(x) = e^{bx - 2\lambda(x - d_H)_+}$$

We then obtain:

$$Z_t' = e^{2\lambda(-d_H)_+}\phi(\widehat{W}_t)\exp(\lambda L_t^{d_H})\exp(-\alpha_+ A_t^{(d_H,+)} - \alpha_- A_t^{(d_H,-)}) \tag{4.13}$$

For the sake of simplicity, we write in this paragraph the strike as Ke^{rT} and need to compute

$$C(t, S_t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\big[(S_T - Ke^{rT})_+ | \mathcal{F}_t\big] \tag{4.14}$$

The pricing formulas[†] are derived in a remarkably simple manner by setting $r = 0$ in the results obtained in sections 4.2.3 and 4.2.4.

[†]All full proofs may be obtained from the authors.

4.3 Numerical Approaches to the NAV option prices

At this point, we are able to compute option prices thanks to Laplace Transforms techniques (see Abate and Whitt (1995)) or Fast Fourier Transforms techniques (see Walker (1996)) . We can observe that if Monte Carlo simulations were performed in order to obtain the NAV option price, the number of such simulations would be fairly large because of the presence of an indicator variable in the Net Asset Value dynamics. The computing time involved in the inversion of Laplace transforms is remarkably lower compared to the one attached to Monte Carlo simulations. The times to maturity considered below are chosen to be less or equal to one year in order to avoid the high water mark reset arising for more distant maturities. Taking into account the reset feature would lead to computations analogous to the ones involved in forward start options and is not the primary focus of this paper.
Tables 1 to 4 show that the call price is an increasing function of the excess performance α, and in turn drift μ, a result to be expected.

The call price is also increasing with the high water mark level H as incentive fees get triggered less often.

Table 5 was just meant to check the exactitude of our coding program : by choosing $a = 0$ and $\alpha = 0$, the NAV call option pricing problem is reduced to the Merton (1973) formula. Table 5 shows that the prices obtained by inversion of the Laplace transform are remarkably close to those provided by the Merton analytical formula.

Table 1

Call Option Prices at a volatility level $\sigma = 20\%$

$H = \$85$, $S_0 = \$100$, $\alpha = 10\%$, $r = 2\%$, $c = 2\%$, $a = 20\%$, $\mu = 15\%$

Strike / Maturity	6 Months	1 Year
90%	\$14.5740	\$18.9619
100%	\$7.6175	\$12.1470
110%	\$3.3054	\$7.2058

$H = S_0 = \$100$, $\alpha = 10\%$, $r = 2\%$, $c = 2\%$, $a = 20\%$ and $\mu = 15\%$

Strike / Maturity	6 Months	1 Year
90%	\$15.0209	\$19.6866
100%	\$7.8346	\$12.5922
110%	\$3.3837	\$7.4427

$H = \$115$, $S_0 = \$100$, $\alpha = 10\%$, $r = 2\%$, $c = 2\%$, $a = 20\%$, $\mu = 15\%$

Strike / Maturity	6 Months	1 Year
90%	\$15.7095	\$20.8464
100%	\$8.4147	\$13.5815
110%	\$3.7084	\$8.1198

Table 2

Call Option Prices at a volatility level $\sigma = 20\%$

$H = \$85$, $S_0 = \$100$, $\alpha = 15\%$, $r = 2\%$, $c = 2\%$, $a = 20\%$, $\mu = 20\%$

Strike / Maturity	6 Months	1 Year
90%	\$16.3804	\$22.6562
100%	\$8.9668	\$15.1925
110%	\$4.1091	\$9.4795

$H = S_0 = \$100$, $\alpha = 15\%$, $r = 2\%$, $c = 2\%$, $a = 20\%$ and $\mu = 20\%$

Strike / Maturity	6 Months	1 Year
90%	\$16.9611	\$23.6036
100%	\$9.2703	\$15.8190
110%	\$4.2276	\$9.8398

$H = \$115$, $S_0 = \$100$, $\alpha = 15\%$, $r = 2\%$, $c = 2\%$, $a = 20\%$, $\mu = 20\%$

Strike / Maturity	6 Months	1 Year
90%	\$17.9362	\$25.2503
100%	\$10.1156	\$17.2719
110%	\$4.7300	\$10.8943

Table 3

Call Option Prices at a volatility level $\sigma = 40\%$

$H = \$85$, $S_0 = \$100$, $\alpha = 10\%$, $r = 2\%$, $c = 2\%$, $a = 20\%$, $\mu = 15\%$

Strike / Maturity	6 Months	1 Year
90%	\$18.8245	\$25.3576
100%	\$13.2042	\$19.9957
110%	\$8.9804	\$15.6276

$H = S_0 = \$100$, $\alpha = 10\%$, $r = 2\%$, $c = 2\%$, $a = 20\%$ and $\mu = 15\%$

Strike / Maturity	6 Months	1 Year
90%	\$19.1239	\$25.8231
100%	\$13.3979	\$20.3534
110%	\$9.1012	\$15.8949

$H = \$115$, $S_0 = \$100$, $\alpha = 10\%$, $r = 2\%$, $c = 2\%$, $a = 20\%$, $\mu = 15\%$

Strike / Maturity	6 Months	1 Year
90%	\$19.5128	\$26.4273
100%	\$13.7277	\$20.8726
110%	\$9.3409	\$16.3134

Table 4
Call Option Prices at a volatility level $\sigma = 40\%$

$H = \$85$, $S_0 = \$100$, $\alpha = 15\%$, $r = 2\%$, $c = 2\%$, $a = 20\%$, $\mu = 20\%$

Strike / Maturity	6 Months	1 Year
90%	\$20.3926	\$28.6499
100%	\$14.4928	\$22.8861
110%	\$9.9903	\$18.1179

$H = S_0 = \$100$, $\alpha = 10\%$, $r = 2\%$, $c = 2\%$, $a = 20\%$ and $\mu = 15\%$

Strike / Maturity	6 Months	1 Year
90%	\$20.7978	\$29.2995
100%	\$14.7618	\$23.3938
110%	\$10.1615	\$18.5044

$H = \$115$, $S_0 = \$100$, $\alpha = 10\%$, $r = 2\%$, $c = 2\%$, $a = 20\%$, $\mu = 15\%$

Strike / Maturity	6 Months	1 Year
90%	\$21.3417	\$30.1555
100%	\$15.2260	\$24.1402
110%	\$10.5042	\$19.1158

Table 5
NAV Call Option Prices when $\mu = 0$ at a volatility level $\sigma = 40\%$
$S_0 = \$100$, $r = 2\%$, $c = 0.3\%$

Maturity	6 months		1 year	
Strike	Laplace Transform	Merton formula	Laplace Transform	Merton formula
90%	\$12.3324	\$12.3324	\$14.577	\$14.577
100%	\$6.0375	\$6.0375	\$8.7434	\$8.7434
110%	\$2.4287	\$2.4287	\$4.8276	\$4.8276

4.4 Conclusion

In this paper, we proposed a pricing formula for options on hedge funds that accounts for the high-water mark rule defining the performance fees paid to the fund managers. The geometric Brownian motion dynamics chosen for the hedge fund Net Asset Value allowed us to exhibit an explicit expression of the Laplace transform in maturity of the option price through the use of Brownian local times. Numerical results obtained by inversion of these Laplace transforms display the influence of key parameters such as volatility or moneyness on the NAV call price.

4.5 Appendix : Excursion Theory

Proof. of Proposition 4.2.5: We use the Master formula exhibited in Brownian excursion theory; for more details see Chapter XII in Revuz and Yor (2005) the notation of which we borrow:
n denotes the Itô characteristic measure of excursions and n_+ is the restriction of n to positive excursions;
$V(\epsilon) = \inf\{t > 0; \epsilon(t) = 0\}$ for $\epsilon \in \mathbf{W}_{exc}$ the space of excursions,
$(\tau_l)_{l \geq 0}$ is the inverse local time of the Brownian motion.

We can write

$$\mathbb{E}\Big[\int_0^\infty dte^{-\frac{\theta}{2}t}h(W_t)\exp(\lambda L_t)\exp(-\mu A_t^+ - \nu A_t^-)\Big] = I \cdot J$$

where

$$\begin{aligned} I &= \mathbb{E}\big[\int_0^\infty dle^{-\frac{\theta}{2}\tau_l}e^{\lambda l}\exp(-\mu A_{\tau_l}^+ - \nu A_{\tau_l}^-)\big] \\ &= \int_0^\infty dl \exp\left(l\big(\lambda - \int n(d\epsilon)(1 - e^{-\frac{\theta}{2}V - \mu A_V^+ - \nu A_V^-})\big)\right) \\ &= \frac{1}{\int n(d\epsilon)\big(1 - e^{-\frac{\theta}{2}V - \mu A_V^+ - \nu A_V^-}\big) - \lambda} \\ &= \frac{1}{\frac{\sqrt{\theta+2\mu}+\sqrt{\theta+2\nu}}{2} - \lambda} \end{aligned}$$

and

$$J = \int_0^\infty dse^{-\frac{\theta}{2}s}\Big\{e^{-\mu s}n_+\big(h(\epsilon_s)\mathbf{1}_{\{s<V\}}\big) + e^{-\nu s}n_+\big(h(-\epsilon_s)\mathbf{1}_{\{s<V\}}\big)\Big\}$$

Next, we use the result

$$n_+\big(\epsilon_s \in dy; s < V\big) = \frac{y}{\sqrt{2\pi s^3}}e^{-\frac{y^2}{2s}}dy \qquad (y > 0) \tag{4.15}$$

and obtain

$$J = \int_0^\infty dxe^{-x\sqrt{\theta+2\mu}}h(x) + \int_0^\infty dxe^{-x\sqrt{\theta+2\nu}}h(-x) \tag{4.16}$$

where the proof of equation (4.16) comes from the fact that in (4.15) the density of n_+ as a function of s, is precisely the density of $T_y = \inf\{t : B_t = y\}$, and $\mathbb{E}[e^{-\lambda T_y}] = e^{-y\sqrt{2\lambda}}$. ∎
This example of application of excursion theory is one of the simplest illustrations of Feynman-Kac type computations which may be obtained with excursion theory arguments. For a more complete story, see Jeanblanc, Pitman and Yor (1997).

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Chapter 5

Correlation and the Pricing of Risks

[*Joint work with Hélyette Geman, Dilip Madan and Marc Yor; published in Annals of Finance, November 2006**]

Given a pricing kernel we investigate the class of risks that are not priced by this kernel. Risks are random payoffs written on underlying uncertainties that may themselves either be random variables, processes, events or information filtrations. A risk is said to be not priced by a kernel if all derivatives on this risk always earn a zero excess return, or equivalently the derivatives may be priced without a change of measure. We say that such risks are not kernel priced. It is shown that reliance on direct correlation between the risk and the pricing kernel as an indicator for the kernel pricing of a risk can be misleading. Examples are given of risks that are uncorrelated with the pricing kernel but are kernel priced. These examples lead to new definitions for risks that are not kernel priced in correlation terms. Additionally we show that the pricing kernel itself viewed as a random variable is strongly negatively kernel priced implying in particular that all monotone increasing functions of the kernel receive a negative risk premium. Moreover the equivalence class of the kernel under increasing monotone transformations is unique in possessing this property.

Keywords Kernel Pricing. Change of Measure. Catastrophic Risk Pricing. Self Sufficient Filtrations.

JEL Classification Numbers G10.G12.G13

5.1 Introduction

An important question in finance is the identification of risks that are priced in the economy. The traditional approach to this issue has been one of ascertaining whether the covariance of returns on financial assets with the risk in question is priced in a classical analysis of cross-sectional excess returns (Fama (1970)). A particular consequence of this method is that risks that are uncorrelated with all financial asset returns, are by virtue of then being untraded, also not priced. However,

*We wish to thank the referee for a very constructive report.

one recognizes that in complete markets there are no nontrivial risks that are uncorrelated with all financial returns. Alternatively, one has the relatively more recent approach developed in the literature on derivatives where an event is priced if the probability used for pricing differs from the statistical or true probability (Harrison and Kreps (1979), Harrison and Pliska (1981)). In the latter case, the Arrow Debreu security associated with this event has a different expectation under the two measures and their ratio reflects the excess return. The two approaches are consistent in recognizing that a risk is priced if it impacts financial excess returns. In particular, we recognize that the latter approach when viewed from the perspective of the former equates the absence of a change of probability, to all functions of the risk in question being uncorrelated with the relevant financial asset return (e.g. the market portfolio in the case of the capital asset pricing model) and hence as earning a zero excess return.

For this paper we define risk as a traded payoff function that is measurable with respect to some information filtration and it is thus a random variable. This payoff random variable however, has a probability law that is deduced from the underlying risks of the payoff. This underlying risk may be as simple as, for example, the level of an equity index at a future point of time. Alternatively it may be a function of the path as is typical for structured products like cliquets or options on the realized variance. The risk in the latter case depends on the law of the process, in contrast with the former case where we focus only on the underlying law of the random variable defined by the level of the index at the specified time. Apart from levels and paths contracts are also written on specific events like corporate defaults or catastrophic losses associated with events like hurricanes or earthquakes. More generally the underlying risk could be that of a subfiltration where we take a $\mathcal{G}$ measurable random variable for a payoff for a general subfiltration $\mathcal{G}$.

For the pricing of risks or underlying risks we shall henceforth refer to a risk as being kernel priced or not kernel priced if the probability law of this risk in question is changed or not changed by a particular candidate pricing kernel or equivalently some contingent claim written on the risk in question earns a nonzero excess return at some time. In this case we say that the underlying risk is priced.

Increasingly one also sees the pricing kernel explicitly defined in a continuous time model, where the risks are described by the natural filtration associated with the time paths of processes. By way of example we cite Bakshi and Chen (1997), Duffie, Pan and Singleton (2000), Chernov and Ghysels (2000), Collin-Dufresne and Goldstein (2002), Carr, Geman, Madan and Yor (2002) and Kimmel (2002). We wish to analyze the precise relation between the absence of a change of probability and concepts of no correlation in such continuous time models with respect to risks defined by various subfiltrations of the economy filtration. For instance one might ask if firm-specific dividend yields in the Bakshi and Chen (1997) model are not kernel priced. Alternatively, and somewhat more interesting is whether the risk of the time to first default in a basket of names is kernel priced by a particular estimated kernel on the larger filtration of the economy. More importantly, it is essential to be careful about incorrectly infering the absence of a change of probability on the mere confirmation of the absence of correlation between a risk and the pricing kernel.

In fact our investigation was initially motivated by comments in the literature on catastrophic risks (see Cummins, Lewis, and Phillips (1999), Froot (1999), Doherty (1997)) where it is argued that reinsurance contracts on losses related to natural disasters should be priced at the actuarial loss value as the risks on such exposures are not correlated with a wide class of financial assets. These reinsurance contracts are derivative contracts and their actuarial valuation implies that they are to be priced without a change of probability or in our terminology they are not kernel priced by any reasonable kernel. It therefore seemed to us that a critical evaluation of the relationship between the absence of correlation and the absence of a change of probability was called for. Our investigation into these questions, for the purpose of rigor and clarity, took an abstract mathematical turn. However, we provide in section 2 below a commentary on the implications of these results to financial questions related to the pricing of contingent claims.

We recognize that a change of probability is intimately linked with the presence of excess returns, positive or negative. We shall show however that, correlation between processes and the pricing kernel, is an important but incomplete guide to issues of kernel pricing. In fact, we provide examples of kernel pricing in situations of zero correlation. A more detailed investigation of the examples reveals that the absence of kernel pricing is related to the absence of correlation provided the latter tests are conducted with respect to a particular set of terminal random variables. These variables define an extended filtration. This filtration includes the time paths of variables useful in predicting the evolution of the risk in question. We call such filtrations *self sufficient* within the economy filtration and define this concept. A filtration may lose the self sufficiency property under a measure change and in this case the absence of kernel pricing is also lost. Hence we study the conditions under which we have the invariance of self sufficiency under changes of probability. Essentially we demonstrate that self sufficiency will be preserved if the pricing kernel does not have any hidden exposure to risks in the self sufficient filtration in a sense made precise later in the paper.

Additionally we show that the pricing kernel, now itself seen as a random variable is “strongly negatively priced” in that the law of the kernel as a random variable, after the measure change first order stochastically dominates its law before the change. As a consequence all monotone increasing functions of the kernel as a random variable or equivalently all random variables aligned with the kernel receive a negative risk premium. Moreover, the kernel (viewed as an equivalence class of random variables under increasing monotone functional transformations) uniquely possesses this property and hence it constitutes the classic insurance asset for the economy for which it is the kernel. The contribution of this paper is threefold: (i) we provide a precise definition of the concept of risks that are not kernel priced, (ii) we demonstrate that the pricing kernel is the unique equivalence class of random variables with the property that all random variables aligned with it receive a negative risk premium, and (iii) we introduce the concept of self sufficient filtration and investigate its role in kernel pricing.

Two sets of correlation tests result. The first addresses no kernel pricing assuming that the chosen self sufficient filtration containing the risk filtration remains self sufficient under the measure change and hence we may evaluate conditional expectations restricted to this chosen self sufficient filtration. These tests deliver the property of almost no kernel pricing abbreviated as $ANKP$. The second collection of tests validate the invariance of self sufficiency under the measure change as required for the full no kernel pricing NKP property. As self sufficient filtrations containing a particular risk filtration are not unique henceforth when we make a reference to ‘the self sufficient filtration’ we mean a fixed particular choice from the collection of self sufficient filtrations.

For $ANKP$ one has to ensure that the projection of the pricing kernel onto the self sufficient filtration containing the risk in question is orthogonal to all the martingales in this self sufficient filtration that finish in a random variable measurable with respect to the risk filtration. This requires that the conditional expectation process of a terminal random variable measurable with respect to the risk filtration, which in general is a martingale adapted to the self sufficient filtration, is orthogonal to the projection of the pricing kernel onto this filtration. This is a fairly large class of orthogonality tests that is well defined in one variable, the projection of the pricing kernel, but is large in the class of terminal random variables to be considered.

For NKP one has to further test that the stochastic logarithm of the kernel less the stochastic logarithm of its projection onto the self sufficient filtration is orthogonal to all the martingales in the self sufficient filtration. This is also a second large class of tests. The deviation between the stochastic logarithm of the kernel and its projection is well defined, but the class of self sufficient martingales is now generated by all the terminal random variables of this filtration which is larger than the risk filtration. We expect that in most cases of practical interest self sufficiency under the physical measure will carry over to self sufficiency under the measure change. The pricing kernel has to be very specially designed to violate self sufficiency under the measure change by constructing an exposure of the kernel to the martingales in the filtration self sufficient under the physical measure that has a null projection onto this filtration and therefore is unobservable to the conditional expectation operator and hence we call such exposures hidden exposures. Barring such hidden exposures in the kernel $ANKP$ is in fact NKP.

The development of econometric procedures for the evaluation of the self sufficiency of a filtration with respect to a larger filtration is an important question for future research into issues of risk pricing proposed by our study. In addition we need to address issues of basis generation of terminal random variables in a filtration to exhaust its martingales and make operative the orthogonality tests proposed here in a relatively abstract framework.

The remainder of the paper is organized as follows. Section 2 presents a financial commentary that draws on the results of the rest of the paper and illustrates their use in addressing issues of interest to the wider financial community. Section 3 briefly addresses the case of a continuous state one period model. The continuous time context with risks defined by the filtration associated with a stochastic process is studied in section 4. We present here our result on the strong negative pricing of the change of measure density, a number of examples connecting correlation and pricing, followed by our result relating the absence of kernel pricing to no correlation. In section 5 we study the important question of the invariance of self sufficiency to a change of measure. Section 6 considers event risk in a continuous time context. Section 7 concludes and presents a few remaining questions, which seem to be hard to solve and should be of interest to theorists in financial economics and probability.

5.2 Financial Commentary

We employ, in this section, the results of this paper to comment on a number of issues related to the pricing of financial risks, a prototype being catastrophic risk. We also comment on the role of self sufficient filtrations in asset pricing in the context of the Heston (1993) model. This latter discussion also illustrates how processes adapted to the pricing kernel may fail to undergo a change of law under the measure change defined by the kernel. These issues are taken up in the following three brief subsections.

5.2.1 Catastrophic Risk and Measure Changes

One may take a stylized view of the effects of catastrophes as driven by a Brownian motion say $(w_1(t), t \geq 0)$ that is orthogonal to a second Brownian motion $(w_2(t), t \geq 0)$ driving the market portfolio or the pricing kernel for that matter. The severity of the impact of catastrophic events depends on things like property values and these may very well be random and related to the market portfolio or the pricing kernel. Hence the integrand of the loss process has a potential dependence on the kernel. The kernel changes very strongly the law of the random variable with which it is associated as shown in Propositions 3 and 4.

Such a view of the situation puts us quite easily in the context of the single period model described by equation (5.2) where we show that there is no reasonable pricing kernel, that is a function of the integrand, that does not change the law of the reinsurance loss process. Hence these considerations suggest that catastrophic loss reinsurance claims should always be priced under a change of measure, notwithstanding claims to the contrary in the existing literature.

5.2.2 Self Sufficiency in the Heston Model

The volatility process of the Heston model is an autonomous Markov process under the stated physical law. It can be shown that the volatility filtration is self sufficient in the full filtration under the physical law. We might ask how this self sufficiency may be lost under the risk neutral law. Or in other words, what sorts of market prices of risk would lead to a loss of self sufficiency under the law induced by the resulting kernel?

It may be shown, using Proposition 6, that as long as the price of volatility risk is just a function of the volatility process, self sufficiency is maintained. We note that all the risk pricing formulations studied in the literature make such an assumption and therefore self sufficiency has been preserved. On the other hand, just adapting the market price of volatility risk to the process for the stock price may not in and of itself involve a loss of self sufficiency. For this the kernel has to have a market price of risk which is orthogonal to the density when projected onto the volatility filtration. Alternatively the market price of volatility risk must have a zero conditional expectation when projected onto the volatility process. Hence there is an exposure that is not observed by the volatility process and that is a specially designed and hidden exposure. We have at this writing not constructed such a hidden exposure in this filtration but expect that it is possible.

5.2.3 Dependence and No change of Measure

The technique of hidden exposures is also the vehicle for ways of building exposures to martingales correlated with the pricing kernel whereby the correlation disappears upon projection onto the self sufficient filtration containing the risk filtration. Since the exposure vanishes upon projection, it guarantees that the risk filtration does not see a change of law even though it was partly adapted to risks correlated with the movements of the kernel.

5.3 Single Period Risk

Consider a one period two date model with uncertainty resolved at time 1 and prices determined at time 0. For simplicity, we suppose that interest rates are zero. The uncertainty in the economy is given by the probability space $(\sum, \mathcal{F}, P)$, where P is the true probability on the event space $(\sum, \mathcal{F})$. Let X denote a random variable defined on this space. This variable could represent a future asset price or the level of some macroeconomic variable.

We wish to consider in addition, a pricing kernel or a candidate for a change of probability. For the current context of a single period zero interest rate economy, it suffices to define it as a positive random variable Y with unit expectation under P. The random variable Y is also commonly referred to in the literature as a state price density (See for example Campbell, Lo and MacKinlay (1997) and the references therein). Hence the pricing measure or risk neutral probability of a set $A \in \mathcal{F}$ is given by the probability Q,

$$Q(A) = E^P \left[Y \mathbf{1}_A \right]$$

where E^P denotes expectation under P and $\mathbf{1}_A$ is the indicator function of the set A.

We define the underlying risk of a random variable X as *not kernel priced* (NKP) by the kernel Y if the distributions of X under P and Q are the same. Equivalently, for any contingent claim paying $f(X)$ at time 1, with finite expectation under P, the risk neutral expectation of $f(X)$ equals its statistical expectation or

$$E^Q[f(X)] = E^P[f(X)]$$

and hence there is no risk premium on all such claims. Note finally that one has

$$NKP \Longleftrightarrow E^P \left[Y|X \right] = 1. \tag{5.1}$$

On the other hand, if the distribution of X under Q differs from that under P then for some positive function f , $0 \leq f \leq 1$ of X, we must have $E^Q[f(X)] < E^P[f(X)]$ and in this case there is a positive risk premium for the security paying $f(X)$ and the contingent claim f represents a risk that is being compensated. Likewise, for the positive function $g(X) = 1 - f(X)$ we have the property $E^Q[g(X)] > E^P[g(X)]$ and the contingent claim g represents a hedge or an insurance contract with a negative excess return. Typically, out-of-the-money puts on the market index, say the S&P500 are examples of the latter while at-the-money index calls are examples of the former (see for example Jackwerth (2000)).

The first simple observation we make, for emphasis, is that the underlying risk of the random variable X may not have the NKP property but yet may be uncorrelated with the kernel Y. Consider for example the case of X a standard normal variable and take for Y the absolute value of X, scaled to a unit expectation. The expectation of the product of Y and X is then zero and we have a zero covariance. However, X and Y are clearly not independent as the latter is just a functional transformation of the former and we observe that the probability distribution of X under Q has been changed. In fact the probability of $X > 1.96$ under P is 2.5% while under Q this probability is easily computed to be above 7%.

On the other hand for an arbitrary kernel Y, if all contingent claims trade and we have a zero covariance for all positive contingent claims so that

$$E^P[f(X)\,(Y(X) - 1)] = 0 \text{ for all } f \geq 0$$

then it must be the case that $Y = 1$, $Q = P$ and there is no change of probability and we have the NKP property. Hence we note that a broader test of orthogonality is linked to the absence of a change of probability. The more general results we develop later for filtrations are in this vein.

Yet another example of zero correlation and a change of probability, related to issues of stochastic volatility, arises when we take the risk X to be of the form

$$X = \sqrt{G}Z$$

where Z is a standard normal variate and G is a positive random variable independent of Z. Here the volatility of the risk conditional on the realization of G is $\sqrt{G}$. To be specific, and to illustrate our point we consider the case when G has the gamma density f_a with parameter a

$$f_a(\gamma) = \frac{\gamma^{a-1}}{\Gamma(a)} e^{-\gamma}$$

then the conditional characteristic function of X given G is

$$E[e^{iuX}|G] = \exp\left(-G\frac{u^2}{2}\right)$$

The unconditional characteristic function is then easily evaluated from the Laplace transform of a gamma variable as

$$E\left[e^{iuX}\right] = \left[\frac{1}{1+\frac{u^2}{2}}\right]^a$$

Now consider the kernel

$$Y = 2^a e^{-G}$$

then Y and X are uncorrelated but the law of X is changed as the Laplace transform of G under this kernel is

$$\begin{aligned} E^Q\left[e^{-\lambda G}\right] &= E^P[2^a e^{-G} e^{-\lambda G}] \\ &= \left[\frac{2}{2+\lambda}\right]^a \end{aligned}$$

It follows that the characteristic function of X under Q is then

$$E^Q[e^{iuX}] = \left[\frac{2}{2+\frac{u^2}{2}}\right]^a.$$

Hence with obvious notation

$$X^Q \overset{(d)}{=} \frac{1}{\sqrt{2}} X^P.$$

Somewhat more generally we may consider the context of two independent random variables, Z a standard normal variate and Y a positive random variable with density $q(y)$, and

$$X = \sqrt{Y}Z. \tag{5.2}$$

In this general context we may show that there is no nontrivial kernel $f(Y)$ for which the law of X is left unchanged. This is done by computing the conditional expectation of $f(Y)$ given X and observing that this could never be 1 for any nonconstant f. To evaluate the conditional expectation we note that for any test function $g(X)$

$$E\left[f(Y)g(X)\right] = \int_{-\infty}^{\infty} dx \int_0^{\infty} dy\ q(y)f(y)\frac{1}{\sqrt{2\pi y}}e^{-\frac{x^2}{2y}}g(x)$$

But we also have with $h(x)$ the density of X that

$$E\left[f(Y)g(X)\right] = \int_{-\infty}^{\infty} dx\ h(x)E\left[f(Y)|X=x\right]g(x)$$

where

$$h(x) = \int_0^{\infty} dy\ q(y)\frac{1}{\sqrt{2\pi y}}e^{-\frac{x^2}{2y}}.$$

Hence it follows that

$$E\left[f(Y)|X=x\right] = \frac{1}{h(x)}\int_0^{\infty} dy\ q(y)f(y)\frac{1}{\sqrt{2\pi y}}e^{-\frac{x^2}{2y}}$$

This conditional expectation is unity just if

$$\int_0^{\infty} dy\ q(y)\frac{1}{\sqrt{2\pi y}}e^{-\frac{x^2}{2y}} = \int_0^{\infty} dy\ q(y)f(y)\frac{1}{\sqrt{2\pi y}}e^{-\frac{x^2}{2y}} \tag{5.3}$$

for all x. We observe that both sides of equation (5.3) are Laplace transforms in $\frac{x^2}{2}$ with respect to the variable $1/y$. By the uniqueness of Laplace transforms we deduce that $f=1$, $q(y)dy$ a.e.

A somewhat different construction of NKP and zero correlation is to take for X an exponential random variable with respect to which the Laguerre polynomials are an orthogonal family and in particular we may use the second Laguerre polynomial $L_2(X) = (X-2)^2 - 2$ to define the change of probability by

$$Y = 1 + \frac{1}{2}L_2(X)$$

that satisfies zero correlation with X but changes the law of X. One could use such constructions to leave invariant the first n moments under the change of measure induced by Y and yet change the law of X.

It is correct to observe that in our examples with zero correlation between X and Y we have induced a dependence for if they were independent then there would be no change of law for X as

$$E^P[Y|X] = E^P[Y] = 1 \tag{5.4}$$

and so we must introduce some dependence and we do. However, it is equally important to note that the NKP property does not mean that Y and X are independent as this requires that for all measurable functions $f, g \geq 0$ we have

$$E[g(Y)f(X)] = E[g(Y)]E[f(X)] \tag{5.5}$$

and the property (5.5) is a much stronger statement than NKP or the property (5.4). In other words it is possible for Y and X to be dependent but yet we have NKP. A simple example is given by

$$Y = e^{XZ-\frac{X^2}{2}} \tag{5.6}$$

where Z is a standard normal variate independent of X a nontrivial (i.e. not equal to a constant almost surely) positive random variable. Clearly Y and X are dependent by construction but as $E[Y|X] = 1$ the NKP condition (5.1) holds.

On the other hand we recognize that if there is a nonzero covariance then the distribution of X under Q is not the same as under P, in fact the mean changes. Furthermore, we note that if we do have a change in probability then there may yet remain contingent claims written on X that have as random variables in their own right the NKP property. This is simply observed by taking again the standard normal variate X under P and changing probability using $Y = c(X^- + \mathbf{1}_{X\geq 0})$ (where c is a normalization constant) which leaves the law of X^+ unchanged. Such a view towards changes in probability has recently been taken in Jarrow and Purnandam (2004).

The general question we wish to investigate is the structure of risks that have the NKP property and those for which the law is changed by a candidate change of measure. A decomposition of the space of risks into those with the NKP property and its complement would be instructive. In particular, is it possible that the set of nontrivial risks with the NKP property is empty or that every nontrivial risk is priced? In the above example of a standard normal distribution for X under P there is always a nontrivial risk with the NKP property. To construct an element of NKP consider the arbitrary change of measure defined by the non constant function $Y(x)$ such that the positive random variable $Y(X)$ has a unit expectation. The function $Y(x)$ must take values both below and above unity and so there exists proper subsets A, B of the real line such that $E[Y(X)|X \in A] < 1$ and $E\left[Y(X)|X \in B\right] > 1$. Let $\mathcal{A} = \{A|E[Y(X)|X \in A] < 1\}$ and let $\mathcal{B} = \{B|E[Y(X)|X \in B] > 1\}$. Let A^* be the union of a maximal chain of elements of $\mathcal{A}$. We must then have that $E\left[Y(X)|X \in A^*\right] = 1$ and as a consequence $E\left[Y(X)|X \in A^{*c}\right] = 1$. Further we note that A^* is a proper subset of the real line. Define the random variable $W = \mathbf{1}_{A^*}$. By construction $E[Y(X)|W] = 1$ and W is a nontrivial risk that has NKP.

For a more concrete example suppose that U is uniform on $[0, 1]$ and $Y(u)$ is $2u$. We can choose $A^* = \left\{u|\frac{1}{4} \leq u \leq \frac{3}{4}\right\}$ and evaluate

$$E\left[Y(u)|u \in A^*\right] = 1.$$

On the other hand for a single period two state model there are no nontrivial risks that are not priced by a measure change that has altered the probability of one of these two states. Essentially there is only one random variable here and it is aligned with the measure change density.

Instead of studying these questions at the level of random variables in a single period context, we recognize that many of the risks that have to be priced in practice involve complex claims referring to time paths of financial or macroeconomic variables. We therefore take up these questions in the more relevant context of continuous time models in the next section.

5.4 Risk in Continuous Time

For our true probability space we consider $\left(\sum, \mathcal{F}_T, \{\mathcal{F}_t\}_{0\leq t\leq T}, P\right)$ a filtered probability space where $\mathcal{F} = (\{\mathcal{F}_t\}_{0\leq t\leq T})$ is the information filtration of the economy. The space of stochastic processes that we shall consider consists of $\mathcal{F}$ semimartingales and the concepts we introduce will be defined for such processes. However, in the construction of examples and counterexamples we shall restrict attention to the widely used model of an $n - dimensional$ Brownian motion $(B(t), t \leq T)$ and its natural filtration. For general results on continuous martingales and Brownian motion we refer to

Revuz and Yor (2001). Well known examples of models in finance formulated with this general structure include the Black and Scholes (1973) and Merton (1973) models for equity options, the jump diffusion model of Merton (1976), the Heath, Jarrow and Morton (1992) model for interest rate risk, the Heston (1993) stochastic volatility model, and more recently the Lévy process models of Eberlein, Keller and Prause (1998) and Carr, Geman, Madan and Yor (2002).

The results of Harrison and Kreps (1979), and Harrison and Pliska (1981) establish the foundations of arbitrage free pricing showing that such prices in a zero interest rate economy are given by expected cash flows evaluated under a change of measure to Q. This change of measure is always given by a positive $\mathcal{F}_T$ measurable random variable D_T that defines the terminal value of the density of the pricing measure Q with respect to the measure P. The change of measure density process $D = (D_t, 0 \leq t \leq T)$ is given by the $(\mathcal{F}, P)$ martingale

$$D_t = E^P \left[D_T | \mathcal{F}_t \right].$$

This density is explicitly modeled in many of the papers already cited and here we reference by way of examples Duffie, Pan and Singleton (2000), Chernov and Ghysels (2000) and Carr, Geman, Madan and Yor (2002). In this context we wish to describe the processes that are kernel priced and those that are not kernel priced and hence have the NKP property at the process level. Specifically we suppose that the pricing kernel is given and may also have been estimated.

Consider for this purpose a set of underlying risks that are specified by an $\mathcal{F}$ semimartingale vector $X = (X_t, 0 \leq t \leq T)$. In general we are interested in the valuation of claims written on the price paths of a set of risk variables and so we consider in general a $\mathcal{X}_T$ measurable functional where $\mathcal{X}_T = \sigma \{X(t), t \leq T\}$ and denote this path functional by $F = F(X(u), u \leq T)$. We also denote by $\mathcal{X}$ the filtration $\mathcal{X} = \{\mathcal{X}_t, t \leq T\}$ where $\mathcal{X}_t = \sigma \{X(s), s \leq t\}$. The prices of contingent claims are developed in a no arbitrage framework by evaluating discounted expected cash flows under a change of probability. For the purposes of this paper we have assumed a zero interest rate environment and hence we consider just the expected cash flow under a change of probability to Q.

At the initial date, prices $\pi_0(F)$ will be given by

$$\pi_0(F) = E^Q \left[F(X(u), u \leq T) | \mathcal{F}_0 \right].$$

At subsequent times $s < T$ a conditional expectation is involved in determining the price. In general we condition on all information at time s (e.g. $\mathcal{F}_s$). We then write

$$\pi_s(F) = E^Q \left[F(X(u), u \leq T) | \mathcal{F}_s \right].$$

Consider now the expectation of F at time s evaluated without a measure change. We write this as

$$\mu_s(F) = E^P \left[F \left(X(u), u \leq T \right) | \mathcal{F}_s \right]$$

However, we now note that with respect to claims that are $\mathcal{X}_T$ measurable there is in general a lot of information in $\mathcal{F}_s$ that has nothing to do with the evolution of the paths of $X(t)$. For a particular example, we know that the level of the stock price at time s has no relevance for the evolution of volatility in the Heston (1993) stochastic volatility model under the specified physical probability P. These considerations lead us to introduce the concept of self sufficient filtrations that we now define.

A subfiltration $\mathcal{J}$ is said to be *self sufficient* in $\mathcal{F}$ if every $(\mathcal{J}, P)$ martingale is an $(\mathcal{F}, P)$ martingale. This condition is equivalent to the statement that for every $J_T \geq 0$, $\mathcal{J}_T$ measurable random variable and $s < T$ we have that

$$E^P [J_T|\mathcal{F}_s] = E^P [J_T|\mathcal{J}_s], \tag{5.7}$$

or equivalently that $\mathcal{J}_T$ and $\mathcal{F}_s$ are conditionally independent given $\mathcal{J}_s$ (Brémaud and Yor (1978)).

Hence for self sufficient filtrations, all information necessary for predicting a $\mathcal{J}_T$ measurable random variable is already in $\mathcal{J}_s$. The concept of self sufficiency was studied in Brémaud and Yor (1978) where it was referred to as hypothesis H. It was later referred to as $\mathcal{J}$ being immersed in $\mathcal{F}$ in Tsirelson (1998) and Emery (2000). For some examples of a lack of self sufficiency involving Brownian motion see Jeulin (1996). We note here that if $\{\mathcal{B}_t\}$ denotes the one-dimensional Brownian filtartion generated by $(B_t, t \geq 0)$, then this filtration loses self sufficiency with respect to any non-trivially enlarged filtration, that is: $\{\mathcal{B}_t \vee \sigma(X)\}_{t\geq 0}$, for X a non constant $\mathcal{B}_\infty - measurable$ variable. Indeed, if (B_t) remained a martingale with respect to $\{\mathcal{B}_t \vee \sigma(X)\}$, then it would be a Brownian motion in that filtration and therefore would be independent of the information at the origin of time, that is $\sigma(X)$, which is absurd since X is a nontrivial functional of (B_t). For a further reference on initial enlargements we mention Mansuy and Yor (2006).

For evaluating conditional expectations we may without loss of generality focus attention on self sufficient filtrations. We denote by $\mathcal{Z}$ any filtration containing $\mathcal{X}$ with the property that $\mathcal{Z}$ is self sufficient in $\mathcal{F}$. We always assume that $\mathcal{Z}_0$ is trivial. The expectation of the claim F at time s is then

$$\begin{aligned} \mu_s(F) &= E^P [F(X(u), u \leq T)|\mathcal{F}_s] \\ &= E^P [F(X(u), u \leq T)|\mathcal{Z}_s] \end{aligned}$$

We say that X has the NKP property with respect to D if for all functionals of the paths of X , $F = F(X(u)_{0\leq u\leq T})$ it is the case that for all $s < T$

$$\pi_s(F) = \mu_s(F) \tag{5.8}$$
$$E^Q [F(X(u), 0 \leq u \leq T)|\mathcal{F}_s] = E^P [F(X(u), 0 \leq u \leq T)|\mathcal{Z}_s] \tag{5.9}$$

We note that expected values are computed in accordance with the right hand side of (5.9). The price is given by the left hand side of (5.9). When NKP holds, all contingent claims written on the paths of X have a zero excess return continuously through time when the pricing kernel is D_T. In this case all the X contingent claims are not priced in the sense of earning zero excess returns continuously through time for investments over arbitrary horizons on any of the derivative securities. The equivalence of (5.9) with (5.8) is based on exploting the self sufficiency of $\mathcal{Z}$ in $\mathcal{F}$ as we may then replace conditioning by $\mathcal{Z}_s$ on the right hand side of equation (5.9) by conditioning on $\mathcal{F}_s$. We make a detailed study of this NKP property (5.8) in sections 4.2 and 5 of this paper.

For ease of notation we shall delete the superscript P on the expectation operator and we understand that all expectations, when no measure is indicated, are under the measure P.

We now provide an important necessary condition for X to possess the NKP property under the additional hypothesis that $\mathcal{X}$ is itself self sufficient in $\mathcal{F}$.

Proposition 1. *If X has the NKP property and $\mathcal{X}$ is itself self sufficient in $\mathcal{F}$ under P then*

$$E\left[D_T|\mathcal{X}_T\right] = 1.$$

where the expectation here is under the measure P.

Proof. Suppose X has the NKP property with respect to D_T. We apply the property (5.9) to the claim given by $F = E[D_T|\mathcal{X}_T]$ to deduce on noting $\mathcal{Z}_s = \mathcal{X}_s$ that

$$E\left[D_T E[D_T|\mathcal{X}_T]|\mathcal{F}_s\right] = E\left[E[D_T|\mathcal{X}_T]|\mathcal{X}_s\right] \tag{5.10}$$

It follows in particular that the left hand side is now $\mathcal{X}_s$ measurable and so

$$\begin{aligned} E\left[D_T E[D_T|\mathcal{X}_T]|\mathcal{F}_s\right] &= E\left[D_T E[D_T|\mathcal{X}_T]|\mathcal{X}_s\right] \\ &= E\left[\left(E\left[D_T|\mathcal{X}_T\right]\right)^2|\mathcal{X}_s\right] \end{aligned} \tag{5.11}$$

Combining equations (5.10) and (5.11) we deduce that

$$E\left[\left(E\left[D_T|\mathcal{X}_T\right]\right)^2|\mathcal{X}_s\right] = E\left[E[D_T|\mathcal{X}_T]|\mathcal{X}_s\right]$$

Taking expectations on both sides, we get:

$$\begin{aligned} E\left[\left(E\left[D_T|\mathcal{X}_T\right]\right)^2\right] &= E\left[E[D_T|\mathcal{X}_T]\right] \\ &= E[D_T] = 1 \\ &= \left(E\left[E[D_T|\mathcal{X}_T]\right]\right)^2 \end{aligned}$$

Hence $E[D_T|\mathcal{X}_T]$ must be a constant equal to its expectation that is unity. ■

This necessary condition for the NKP property when $\mathcal{X}$ is self sufficient in $\mathcal{F}$ does not on its own imply the NKP property. From this necessary condition we may deduce that

$$\begin{aligned} E^Q\left[F\left(X(u), u \le T\right)|\mathcal{X}_s\right] &= E\left[D_T F\left(X(u), u \le T\right)|\mathcal{X}_s\right] \\ &= E\left[E\left[D_T|\mathcal{X}_T\right] F\left(X(u), u \le T\right)|\mathcal{X}_s\right] \\ &= E\left[F\left(X(u), u \le T\right)|\mathcal{X}_s\right] \end{aligned} \tag{5.12}$$

We then get the NKP property provided we also have

$$E^Q\left[F\left(X(u), u \le T\right)|\mathcal{F}_s\right] = E^Q\left[F\left(X(u), u \le T\right)|\mathcal{X}_s\right] \tag{5.13}$$

which is precisely the condition that $\mathcal{X}$ is self sufficient in $\mathcal{F}$ under Q.

Hence we may state the result

Proposition 2 *If X is self sufficient in F under both P and Q then the NKP property holds if and only if*

$$E\left[D_T|\mathcal{X}_T\right] = 1 \tag{5.14}$$

These considerations motivate our interest in the property (5.14) as a vehicle in understanding the NKP property. In addition we present explicit results showing how self sufficiency may be maintained under a change of probability.

We may usefully decompose the full NKP property into two properties. The first we call the almost NKP property, $ANKP$, obtained via conditioning on the left hand side of equation (5.9) by $\mathcal{Z}_s$ in place of $\mathcal{F}_s$. The second property is then the self sufficiency of $\mathcal{Z}$ in $\mathcal{F}$ under Q or equation (5.13). This is studied later in section 5 where we observe that except for specially designed kernels with what we call hidden exposures to the self sufficient filtration the self sufficiency is preserved under Q and we do in fact have the NKP property. In the continuous process context these hidden exposures are stochastic integrals with respect to martingales in the self sufficient filtration that have integrands with zero conditional expectation and hence we say the exposures are hidden to the self sufficient filtration.

We thus define X to have the $ANKP$ property with respect to D if there exists a filtration $\mathcal{Z}$ which is self sufficient in $\mathcal{F}$ under P containing $\mathcal{X}$ such that for all functionals of the paths of X , $F = F((X(u)_{0 \le u \le T}))$ it is the case that for all $s < T$

$$E^Q \left[F(X(u)_{0 \le u \le T}) | \mathcal{Z}_s \right] = E^P \left[F(X(u)_{0 \le u \le T}) | \mathcal{Z}_s \right] \quad (5.15)$$
$$E^P \left[DF(X(u)_{0 \le u \le T}) | \mathcal{Z}_s \right] = E^P \left[F(X(u)_{0 \le u \le T}) | \mathcal{Z}_s \right] \quad (5.16)$$

The condition (5.16) is equivalent to the assertion that there is no change of law of the process X on the filtration $\mathcal{Z}$. But as the probability law depends on the filtration there may be a change of law on the larger filtration $\mathcal{F}$. In addition to $ANKP$ we require the self sufficiency of $\mathcal{Z}$ in $\mathcal{F}$ under Q to lift the absence of a change of probability on $\mathcal{Z}$ to the absence of such a change of probability on $\mathcal{F}$. We further note by Proposition 1 and a computation similar to (5.12), that the condition $E[D_T|\mathcal{X}_T] = 1$ is both necessary and sufficient for the $ANKP$ property in the case $\mathcal{X}$ is itself self sufficient in $\mathcal{F}$.

Finally we note that the structure of risks studied need not be associated with the continuous time paths of processes. One could be interested in other filtrations of the economy. We might consider for example just the set of times at which the stock markets attain a new peak and then extract the filtration of the economy at these times. Such a sequence of filtrations is no longer associated with the path of a process in continuous time. We may then ask whether the NKP property holds for such a filtration. In case it does, then derivatives written on the market index with payoffs measurable with respect to its peaks would be priced with no change of probability.

Hence we define more generally for an arbitrary increasing filtration $\mathcal{G} = \{\mathcal{G}_t, t \le T\}$, that $\mathcal{G}$ is *not kernel priced* or has the NKP property with respect to D if for $s \le T$, $G_T \ge 0$,and G_T is $\mathcal{G}_T$ measurable

$$E^Q \left[G_T | \mathcal{F}_s \right] = E^P \left[G_T | \mathcal{Z}_s \right] \quad (5.17)$$

where $\mathcal{Z}$ is a filtration containing $\mathcal{G}$ and $\mathcal{Z}$ is self sufficient in $\mathcal{F}$.

Similarly we define that $\mathcal{G}$ has the *almost not kernel priced property* ($ANKP$) with respect to D if for $s \le T$, $G_T \ge 0$,and G_T is $\mathcal{G}_T$ measurable

$$E^Q \left[G_T | \mathcal{Z}_s \right] = E^P \left[G_T | \mathcal{Z}_s \right] \quad (5.18)$$

where $\mathcal{Z}$ is a filtration containing $\mathcal{G}$ and $\mathcal{Z}$ is self sufficient in $\mathcal{F}$.

Apart from the not kernel priced or NKP risks, we are also interested in the set of risks that are definitely priced. We next define the set of risks that are strongly kernel priced SKP and strongly negatively kernel priced $SNKP$.

We say that X is *strongly kernel priced* or has the property SKP if for all n , $s < T$ and times $t_1 < t_2 < \cdots < t_n \leq T$ and arbitrary levels $a_1, a_2, \cdots a_n$

$$Q\left((X(t_i) < a_i)_{1 \leq i \leq n} | \mathcal{F}_s\right) > P\left((X(t_i) < a_i)_{1 \leq i \leq n} | \mathcal{F}_s\right). \tag{5.19}$$

For a strongly priced risk, in particular the conditional marginal cumulative distribution functions have been lifted up. Such a condition in the univariate case is well known to be equivalent to the statement that the law of X under P first order stochastically dominates the law under Q. Equivalently we have the property that every monotone increasing function of a strongly priced risk has a higher expectation under P than under Q and hence a positive risk premium. We call such monotone increasing functions of a risk X, risks that are aligned with X. For strongly priced risks all aligned risks have a positive risk premium.

Alternatively, we say that X is *strongly negatively kernel priced* or has the $SNKP$ property if $-X$ has the SKP property. For strongly negatively priced risks all aligned random variables have a negative risk premium.

We first show that the density D_T itself has the $SNKP$ property. Furthermore, the equivalence of random variables aligned with the kernel is unique in possessing this property. Hence the acquisition of cash flows aligned with the pricing kernel are the classic insurance contracts of the economy. This observation is in line with basic economic principles that indicate a relatively over weighting of states with the worse outcomes in measure changes. However, here we see these properties at a more general level without any appeal to utility theory. These are the considerations that lead to a basic representation of excess returns in terms of covariations of returns with the negative of the growth rate in the pricing density (See Back (1991)).

Proposition 3 *D_T has the $SNKP$ property.*

Proof. We may observe that D_T is $SNKP$ by first considering $n = 1$, $s < t \leq T$ and $a < D_s$ for which

$$\begin{aligned} Q\left[(D(t) < a)|\mathcal{F}_s\right] &= \frac{E^P\left[\mathbf{1}_{D(t)<a} D_T | \mathcal{F}_s\right]}{D_s} \\ &= \frac{E^P\left[\mathbf{1}_{D(t)<a} D_t | \mathcal{F}_s\right]}{D_s} \\ &< E^P[\mathbf{1}_{D(t)<a} | \mathcal{F}_s] \end{aligned}$$

where the last inequality follows on noting that on the set $D(t) < a$, $\frac{D(t)}{D(s)} < \frac{a}{D(s)} < 1$ by the choice of a. On the other hand for $s < t \leq T$ and $a > D(s)$ we have

$$\begin{aligned} Q\left[(D(t) > a)|\mathcal{F}_s\right] &= \frac{E^P\left[\mathbf{1}_{D(t)>a} D_T | \mathcal{F}_s\right]}{D_s} \\ &= \frac{E^P\left[\mathbf{1}_{D(t)>a} D_t | \mathcal{F}_s\right]}{D_s} \\ &> E^P[\mathbf{1}_{D(t)>a} | \mathcal{F}_s]. \end{aligned}$$

Hence all the conditional cumulative marginals distribution functions for $D(t)$ have been shifted downwards. This argument may be generalized by induction to show that D_T has the $SNKP$ property. ■

The result that the marginal cumulative distributions of D_T under Q are shifted down is related to the Hardy-Littlewood result (See Dubins and Gilat (1978)) that

$$v(x) = E\left[X|X > x\right]$$

is an increasing function of x. For X a positive change of measure density the function v is unity at 0 and hence is above unity over its range. However it is also the ratio of the probability that $(X > x)$ under Q to that under P.

Proposition 4 *D_T is the only equivalence class of random variables (up to increasing monotonic transformations) with the property that every monotone increasing function of the random variable has a negative risk premium.*

Proof. Consider a random variable Γ that is not aligned with the density. Define by C the set of all random variables aligned with Γ. The set C contains all random variables that are monotone increasing functions of Γ and we easily see that C is a convex set and $D_T - 1 \notin C$. By convex set separation we may obtain a random variable Y such that

$$E[Y(D_T - 1)] < 0 \tag{5.20}$$

and for all $X \in C$ we have

$$E\left[YX\right] \geq 0.$$

It follows from equation (5.20) that Y has a positive risk premium. Furthermore if Y is not aligned with Γ then one may construct X aligned with Γ for which $E[XY] < 0$ and so $Y \in C$ and we have a variable aligned with Γ with a positive risk premium. ■

5.4.1 The Examples

We now investigate further the NKP properties with respect to a specific D. Our first example supports the basic intuition of considering correlations in making judgements about the kernel pricing of risks or about the NKP property.

Example 1 *If $X = (X_t, 0 \leq t \leq T)$ is a $(\mathcal{F}, P)$ one dimensional Brownian motion and X, D are orthogonal as $(\mathcal{F}, P)$ martingales then X has the NKP property with $\mathcal{Z} = \mathcal{F}$.*

Proof. The orthogonality assumption implies that X is a Q local martingale, and since the quadratic variation of X under Q is the same as that under P by Girsanov's theorem we have that

$$< X >_t^Q = < X >_t^P = t,$$

and it follows by Lévy's theorem that X is a $(\mathcal{F}, Q)$ Brownian motion; in particular, (5.8) holds. ■

For a particular illustration of example 1, let $B = (B_t, 0 \leq t \leq T)$ be a n dimensional Brownian motion with $n > 1$ and let $\mathcal{F} = \left\{\sigma\left\{B_s, s \leq t\right\}, t \leq T\right\}$. Then we know that:

$$D(t) = 1 + \int_0^t \delta_s \cdot dB_s$$

for some $\mathcal{F}$ predictable process δ_s such that $\int_0^t |\delta_s|^2 ds$ is almost surely finite, and that likewise:

$$X(t) = X(0) + \int_0^t x_s \cdot dB_s$$

with our assumption implying here that

$$\begin{aligned}
|x_s|^2 &= 1, \ dsdP \text{ almost surely} \\
x_s \cdot \delta_s &= 0.
\end{aligned}$$

Even more specifically we could take $n = 2$, $B = (U, V)$ a two dimensional standard Brownian motion and

$$\begin{aligned}
D(t) &= \exp\left(A(t) - \frac{t}{2}\right) \\
A(t) &= \int_0^t \frac{U(s)dU(s) + V(s)dV(s)}{\sqrt{U(s)^2 + V(s)^2}} \\
X(t) &= \int_0^t \frac{-V(s)dU(s) + U(s)dV(s)}{\sqrt{U(s)^2 + V(s)^2}}
\end{aligned}$$

This example illustrates the type of considerations involved in widely used continuous time models supporting correlation calculations as fundamental to evaluating the kernel pricing of risk. We see here a tight connection between the absence of correlation and the absence of a change of probability or the NKP property. However, the situation can be more subtle even in the standard context of a one dimensional Brownian motion model. This is illustrated in our next example.

Example 2 *We consider here $X = (X_t, t \leq T)$ as a $(\mathcal{F}, P)$ Brownian motion, $\mathcal{X}_T = \sigma\{X_t, t \leq T\}$ with the property that*

$$E^P[D(T)|\mathcal{X}_T] = 1. \tag{5.21}$$

These constructions give us examples of the $ANKP$ property with $\mathcal{Z} = \mathcal{X}$ under P. *In fact we have noted that (5.21) is equivalent to (5.16) when $\mathcal{X}$ is self sufficient in F under P.*

In particular, consider for example in a one dimensional Brownian motion $(B(t), t \leq T)$ context:

$$D(T) = 1 + \lambda sign(B(T)) \tag{5.22}$$

for $|\lambda| < 1$, and

$$X(t) = \int_0^t sign(B(s))dB(s).$$

As is well known, the filtration of X is self sufficient (so $\mathcal{Z} = \mathcal{X}$), and $\mathcal{X}(t) = \sigma\{|B(s)|, s \leq t\}$, hence $\mathcal{X}(T)$ is independent from $sign(B(T))$ which immediately implies that (5.21) holds and hence that

$$Q_{|\mathcal{X}(T)} = P_{|\mathcal{X}(T)}. \tag{5.23}$$

But note that on the filtration $\mathcal{F}$, $X = (X_t, t \leq T)$ is a $(\mathcal{F}, P)$ Brownian motion and not a $(\mathcal{F}, Q)$ Brownian motion and has on $\mathcal{F}$ a nonzero instantaneous correlation structure with the kernel. It follows in particular from the considerations we investigate later in Section 5, that $\mathcal{X}$ is not self sufficient in $\mathcal{F}$ under Q and the $ANKP$ does not lead to NKP in this case.

We mention this case as it is often customary to define a filtration under P describing a set of risks as for example in a stochastic volatility model of the Heston type where it is clear that the filtration describing the evolution of volatility and the stock is self sufficient under P. Now a prospective measure change could leave the law of the stock price and volatility unchanged when attention is restricted to this filtration and so a researcher may be inclined to believe that they have NKP but in fact they may only have $ANKP$ for under Q the market price of volatility risk may be dependent on other variables that are correlated with either the stock or the volatility and these must be included in the self sufficient filtration under Q.

This example may instructively be generalized somewhat by considering an odd functional of Brownian motion, that is:

$$\Phi(B) = \Phi(B(u), u \leq T),$$

such that

$$\begin{aligned} (i)\ ||\Phi||_\infty &< 1 \\ (ii)\ \Phi(B) &= -\Phi(-B) \end{aligned}$$

Now take

$$D(T) = 1 + \Phi(B)$$

It is still true that

$$E^P[D(T)|\mathcal{X}(T)] = 1,$$

as (5.21) is equivalent to (5.24)

$$E^P\left[\Phi(B)|\mathcal{X}(T)\right] = 0, \tag{5.24}$$

and (5.24) follows from the oddity of Φ, condition (ii) above, since for every $F \geq 0$

$$E^P\left[F(|B(u)|, u \leq T)\Phi(B)\right] = -E^P\left[F(|B(u)|, u \leq T)\Phi(B)\right]$$

and hence (5.24).

Although this argument seems very easy, if we are given $D = (D(t), t \leq T)$ in exponential form

$$D(t) = \exp\left(\int_0^t h_s(B)dB(s) - \frac{1}{2}\int_0^t h_s^2(B)ds\right)$$

with $\int_0^t h_s^2(B)ds < \infty$ almost surely for all $t \leq T$, then it may not be so obvious at first sight that (5.23) is satisfied. In fact for the example of equation (5.22) we may use the general formula for every bounded Borel function f, with
$\mathcal{B}(t) = \sigma\left\{B(s), s \leq t\right\},$

$$\begin{aligned} E\left[f(B(T))|\mathcal{B}(t)\right] &= E[f(B(T))] + \int_0^t \frac{\partial V}{\partial B}|_{(s,B(s))} dB(s) \\ V(s,x) &= E\left[f(B(T))|B(s) = x\right] \\ &= \int_{-\infty}^{\infty} dy \frac{\exp\left(-\frac{(y-x)^2}{2(T-s)}\right)}{\sqrt{2\pi(T-s)}} f(y). \end{aligned}$$

For our example of equation (5.22) we have

$$\begin{aligned} V(s,x) &= 1+\lambda\int_{-\infty}^{\infty} dy \frac{\exp\left(-\frac{(y-x)^2}{2(T-s)}\right)}{\sqrt{2\pi(T-s)}} sign(y) \\ &= 1+\lambda\int_{0}^{\infty} dy \frac{\exp\left(-\frac{(y-x)^2}{2(T-s)}\right)}{\sqrt{2\pi(T-s)}} - \lambda\int_{0}^{\infty} dy \frac{\exp\left(-\frac{(y+x)^2}{2(T-s)}\right)}{\sqrt{2\pi(T-s)}} \\ &= 1+\lambda\left(2N\left(\frac{x}{\sqrt{T-s}}\right)-1\right) \end{aligned}$$

where $N(x)$ is the standard normal distribution function. It follows that

$$\begin{aligned} \frac{\partial V}{\partial x} &= \frac{2\lambda}{\sqrt{(T-s)}} n\left(\frac{x}{\sqrt{T-s}}\right) \\ &\underset{Def}{=} q(s,x) \end{aligned}$$

where $n(x)$ is the standard normal density.

To develop the exponential form for $D(t)$ we note that

$$\begin{aligned} D(t) &= V(t,B(t)) \\ &= 1+\int_0^t q(s,B(s))dB(s) \\ &= 1+\int_0^t V(s,B(s))\left(\frac{q(s,B(s))}{V(s,B(s))}\right)dB(s) \\ &= \exp\left(\int_0^t \left(\frac{q(s,B(s))}{V(s,B(s))}\right)dB(s) - \frac{1}{2}\int_0^t \left(\frac{q(s,B(s))}{V(s,B(s))}\right)^2 ds\right) \end{aligned}$$

and we have

$$\begin{aligned} & h(s,B(s)) \\ = \; & \left(1+\lambda\left(2N\left(\frac{B(s)}{\sqrt{T-s}}\right)-1\right)\right)^{-1} \frac{2\lambda}{\sqrt{(T-s)}} n\left(\frac{B(s)}{\sqrt{T-s}}\right) \end{aligned}$$

In general we have

$$\Phi(B) = \int_0^T \phi(s,(B(u),u\le s))dB(s)$$

From the oddity of Φ and the uniqueness of ϕ we deduce that ϕ is even or that

$$\phi(s,(B(u),u\le s)) = \phi(s,-(B(u),u\le s))$$

as is also the case for our example $h(s,B(s))$. Under Q we get

$$\begin{aligned} B(t) &= \widetilde{B}(t) + \int_0^t \frac{\phi(s,(B(u),u\le s))}{1+\Phi(s,(B(u),u\le s))} ds \\ &= \widetilde{B}(t) + \int_0^t \frac{\phi(s,(B(u),u\le s))}{1+E\left[\Phi(B)|\mathcal{F}_s\right]} ds \end{aligned}$$

with $\widetilde{B}$ an $(\mathcal{F}, Q)$ Brownian motion. We may now exhibit the dynamics of X under Q,

$$\begin{aligned} X(t) &= \int_0^t sign(B(s))dB(s) \\ &= \int_0^t sign(B(s))d\widetilde{B}(s) + \int_0^t \frac{sign(B(s))\phi(s,(B(u),u \le s))}{1+\Phi(s,(B(u),u \le s))}ds \end{aligned}$$

so that $X(t)$ is a semimartingale under Q reflecting a nonzero correlation structure but $X(t)$ has the $ANKP$ property. That $X(t)$ is a martingale with respect to its own filtration $\mathcal{X}(t)$ follows from the fact that

$$E^Q\left[\frac{sign(B(s))\phi(s,(B(u),u \le s))}{1+\Phi(s,(B(u),u \le s))}|\mathcal{X}(s)\right] = 0 \tag{5.25}$$

This follows on observing that the left hand side of equation (5.25) is

$$\begin{aligned} \frac{E^P\left[D(s)sign(B(s))|\mathcal{X}(s)\right]}{E^P\left[D(s))|\mathcal{X}(s)\right]} &= E^P\left[sign(B(s))|\mathcal{X}(s)\right] \\ &= 0. \end{aligned}$$

Example 2 illustrates an interesting situation: The existence of a change of measure density that prices positively the risk of the Brownian motion driving the economy continuously through time: Yet there exist martingales correlated with the change of measure density continuously through time that are not priced when attention is restricted to the risk filtration itself. It is true that in this example the expected correlation projected onto the filtration $\mathcal{X}(s)$ of the martingale is zero by construction, but in practice such a situation is not that easy to detect.

We grant that example 2 is one dimensional but a comparable discussion may easily be embedded into a higher dimensional context. For example we could consider the Heston model for the evolution of the stock price $S(t)$ and the volatility $v(t)$. Now consider a pricing kernel of the form

$$D(T) = 1 + \lambda sign(B(T))F(S(T), v(T)) \tag{5.26}$$

for a bounded and positive function $F(S(T), v(T))$ with $B(t), t > 0$ an independent Brownian motion. Let $\mathcal{X}$ be the filtration of the stock and its volatility. For any $\mathcal{X}_T$ measurable random variable C_T we have that

$$\begin{aligned} E^P\left[C_T D(T)\right] &= E^P[E^P\left[C_T D(T)|\mathcal{X}_T\right]] \\ &= E^P[C_T E^P[D(T)|\mathcal{X}_T]] \\ &= E^P[C_T]E^P[D(T)] \end{aligned}$$

and we have no correlation between the risk and the density as random variables at time zero. We also have no correlation between the risk and the density as random variables at all times $s > 0$ with respect to the risk filtration as

$$\begin{aligned} E^P\left[C_T D(T)|\mathcal{X}_s\right] &= E^P[E^P\left[C_T D(T)|\mathcal{X}_T\right]|\mathcal{X}_s] \\ &= E^P[C_T E^P[D(T)|\mathcal{X}_T]|\mathcal{X}_s] \\ &= E^P[C_T|\mathcal{X}_s]E^P[D(T)|\mathcal{X}_s] \end{aligned}$$

But the filtration $\mathcal{X}$ is not self sufficient in $\mathcal{F}$ under Q and we do have correlation between the risk and the density at time $s > 0$ with respect to the filtration $\mathcal{F}_s$ as

$$E\left[C_T D(T)|\mathcal{F}_s\right] = E^P[E^P\left[C_T\left(1+\lambda\left(2N\left(\frac{B(s)}{\sqrt{T-s}}\right)-1\right)F(S(T),v(T))\right)\right]|\mathcal{F}_s]$$

where the expression in brackets multiplying λ is just the conditional expectation of $sign(B(T))$ conditional on $B(s)$. We could now take by way of example for C_T the variable $F(S(T),v(T))$ to get correlation conditional on $\mathcal{F}_s$. Here NKP fails but $ANKP$ holds. By construction we have given the density an exposure to movements in the stock and volatility process that is not visible to the risk filtration and projects to zero.

To see the hidden nature of the exposure in the density (5.26) one explicitly writes out the $(\mathcal{F}, P)$ martingale associated with the density (5.26) as a terminal random variable as a stochastic integral with respect to the Brownian motions $(B(t), t > 0)$, and the two Brownian motions $(W_1(t), t > 0)$ and $(W_2(t), t > 0)$ that drive the stock and the volatility processes. Assuming that one has written $F(S(T), v(T))$ in the form

$$F\left(S(T),v(T)\right) = E\left[F(S(T),v(T))\right] + \int_0^T \left(f_1(s)dW_1(s) + f_2(s)dW_2(s)\right)$$

for some integrands f_1, f_2 one may show that the exposure of the density to the filtration generated by the stock and volatility processes takes the form

$$\begin{aligned} D(T) &= 1 + \int_0^T a(s)dB(s) + \int_0^T E\left[\lambda sign(B(T)|\mathcal{B}_s\right]\left((f_1(s)dW_1(s) + f_2(s)dW_2(s))\right) \\ &= 1 + \int_0^T a(s)dB(s) + \int_0^T \lambda\left(2N\left(\frac{B(s)}{\sqrt{T-s}}\right)-1\right)\left((f_1(s)dW_1(s) + f_2(s)dW_2(s))\right) \end{aligned}$$

but as the function

$$2N\left(\frac{B(s)}{\sqrt{T-s}}\right)-1$$

is an antisymmetric function of $B(s)$ that is independent of the stock and volatility processes, its projection onto this filtration is zero. The exposure of the density to the stock and volatility filtration is therefore hidden from this filtration.

Our next example illustrates the possibility in one dimension of a situation where we have zero correlation between a risk and the change of measure but the risk in question is kernel priced and does not have the NKP property.

Example 3 *Define the density process by*

$$D(t) = \exp\left(\int_0^t \mathbf{1}_{B(s)>0}dB(s) - \frac{1}{2}\int_0^t \mathbf{1}_{B(s)>0}ds\right)$$

and let $X = (X(t), t \leq T)$ *be the martingale*

$$X(t) = \int_0^t \mathbf{1}_{B(s)<0}dB(s).$$

The law of X *is changed by the density* D.

Proof. It is clear by construction that

$$\begin{aligned}\langle X, D\rangle_t &= \int_0^t D(s)\mathbf{1}_{B(s)>0}\mathbf{1}_{B(s)<0}ds \\ &= 0\end{aligned}$$

and there is no correlation between the density D and the martingale X.

However the quadratic variation of X is

$$\begin{aligned}\langle X\rangle_t &= \int_0^t \mathbf{1}_{B(s)<0}ds \\ &= t - \int_0^t \mathbf{1}_{B(s)>0}ds\end{aligned}$$

and the law of this quadratic variation under Q is different from its law under P. More specifically:

$$E^Q\left[\mathbf{1}_{(B(t)<0)}\right] < \frac{1}{2} = E^P\left[\mathbf{1}_{(B(t)<0)}\right].$$

Indeed

$$\begin{aligned}E^Q\left[\mathbf{1}_{(B(t)<0)}\right] &= E^P\left[\mathbf{1}_{(B(t)<0)}D(t)\right] \\ &= E^P\left[\mathbf{1}_{(B(t)<0)}\exp\left(-\frac{1}{2}\Sigma(t) - \frac{1}{2}\int_0^t \mathbf{1}_{(B(s)>0)}ds\right)\right]\end{aligned}$$

since on $(B(t) < 0)$, $L(t) = \int_0^t \mathbf{1}_{B(s)>0}dB(s) \equiv B(t)^+ - \frac{1}{2}\Sigma(t) = -\frac{1}{2}\Sigma(t)$ where $\Sigma(t)$ is the local time of Brownian motion at zero. ■

Another example of zero correlation with absence of NKP property is provided by the following two dimensional construction.

Example 4 *Let $\mathbb{B} = (U, V)$ be a standard two dimensional Brownian motion. Define*

$$\begin{aligned}R^2(t) &= U^2(t) + V^2(t) \\ D(t) &= \exp\left(\int_0^t U(s)dU(s) + V(s)dV(s) - \frac{1}{2}\int_0^t R^2(s)ds\right) \\ &= \exp\left(\frac{1}{2}(R^2(t) - 2t) - \frac{1}{2}\int_0^t R^2(s)ds\right)\end{aligned}$$

Further let

$$X(t) = \int_0^t -U(s)dV(s) + V(s)dU(s)$$

then X is a martingale under both P and Q but X does not have the NKP property.

Proof. We easily verify that X remains a martingale under Q as

$$\begin{aligned}\langle X, D\rangle_t &= \int_0^t D(s)(-U(s)V(s) + U(s)V(s))ds \\ &= 0.\end{aligned}$$

But the probability law of X under Q is not that under P. In fact both under P and Q we have that

$$X(t) = \gamma \left(\int_0^t R^2(s)ds \right)$$

where $(\gamma(u), u \geq 0)$ is a one dimensional Brownian motion independent of R, but the law of R (hence the law of $\langle X \rangle$) is different under P and Q.

Under Q we have that

$$\begin{aligned} U(t) &= \widetilde{U}(t) + \int_0^t U(s)ds \\ V(t) &= \widetilde{V}(t) + \int_0^t V(s)ds \end{aligned}$$

where $\left(\widetilde{U}, \widetilde{V}\right)$ is a two dimensional $(\mathcal{F}, Q)$ Brownian motion; hence, under Q, U and V are two independent Ornstein-Uhlenbeck processes, and

$$R^2(t) = 2\int_0^t R(s)d\widetilde{\beta}(s) + 2\int_0^t R^2(s)ds + 2t$$

with $\widetilde{\beta} = (\widetilde{\beta}(t), t \geq 0)$ a Brownian motion, is a CIR (Cox, Ingersoll, Ross (1985)) type process. While under P we have that

$$R^2(t) = 2\int_0^t R(s)d\beta(s) + 2t,$$

for a Brownian motion $\beta = (\beta(t), t \geq 0)$.We note that under the respective probabilities P and Q, the Brownian motions $\widetilde{\beta}, \beta$ are independent of γ. ■

We have now seen examples of nonzero correlation and presence of the $ANKP$ property and examples of zero correlation and absence of the NKP property. From the perspective of financial economics and valuation of claims, the NKP property is fundamental. An analysis of the $ANKP$ property in terms of orthogonality is precisely developed in subsection 4.2 below. This is followed in section 5 by a study of the invariance of self sufficiency to the measure change, the property needed to infer NKP from $ANKP$. We now illustrate the case of a risk satisfying $ANKP$ with respect to the simplest measure change in the one-dimensional Brownian context.

Example 5 *Consider the change of measure density*

$$D(t) = \exp\left(\lambda B(t) - \frac{\lambda^2 t}{2}\right)$$

so that under Q, $B(t)$ is a Brownian motion with drift λ. Let

$$X(t) = B(t) - \int_0^t \frac{B(s)}{s} ds.$$

The semimartingale $X = (X(t), t \geq 0)$ is a Brownian motion both under P and Q.

Proof. Let $\mathcal{G}(t) = \sigma\{X(u), u \leq t\}$. We verify that in its own filtration the process X is Brownian motion. It is clear that it is a Gaussian process with zero mean and covariance function

$$C(s,t) = E^P\left[\left(B(t) - \int_0^t \frac{B(u)}{u}du\right)\left(B(s) - \int_0^s \frac{B(v)}{v}dv\right)\right]$$

and for $s < t$ we have

$$\begin{aligned}
C(s,t) &= s - E^P\left[\int_0^s \frac{B(u)B(s)}{u}du + \int_s^t \frac{B(u)B(s)}{u}du\right] \\
&\quad -E^P\left[\int_0^s \frac{B(t)B(v)}{v}dv\right] \\
&\quad +E^P\left[\int_0^s \int_0^s \frac{B(u)B(v)}{uv}dvdu + \int_s^t \int_0^s \frac{B(u)B(v)}{uv}dvdu\right] \\
&= s - s - s\log\left(\frac{t}{s}\right) - s + \int_0^s \int_0^s \frac{u \wedge v}{uv}dvdu \\
&\quad +s\log\left(\frac{t}{s}\right) \\
&= -s + \int_0^s du \int_0^u dv\frac{1}{u} + \int_0^s du \int_u^s \frac{1}{v}dv \\
&= -s + s + \int_0^s dv \int_0^v \frac{1}{v}du \\
&= s.
\end{aligned}$$

We therefore have that

$$E\left[D(t)|\mathcal{G}(t)\right] = 1$$

In fact we have that

$$E[B(t)X(s)] = 0$$

for $s \leq t$ and $B(t)$ is independent of $\mathcal{G}(t)$. ■

We see here that even the simplest measure change that prices the Brownian motion at a constant price continuously through time, fails to change probability when restricted to the filtration of the risk for some semimartingale risks. We note however, that though probability has not been changed we do not here have the NKP property as the filtration of X is not self sufficient in $\mathcal{F}$ under either P or Q.

The result that $X(t)$ in example 5 is a Brownian motion is explained in greater detail in chapter 1 of Yor (1992). It is obtained by considering the semimartingale decomposition of Brownian motion $(B(u), u \leq t)$ enlarged by its value $B(t)$ at time t, and then reversing time from time t.

We further comment that $X(t)$ in example 5 has the $ANKP$ property with respect to any change of measure density $D(t) = h(B(t), t)$ where h is a positive space time harmonic function which is known to be of the form

$$h(B(t),t) = \int_{-\infty}^{\infty} \mu(d\lambda)\exp\left(\lambda B(t) - \frac{\lambda^2 t}{2}\right)$$

for any probability measure μ on $\mathbb{R}$.

5.4.2 Almost No Kernel Pricing for filtrations

In this subsection we clarify precisely what the $ANKP$ property means in terms of orthogonality relations between certain martingales. For this purpose we discuss the $ANKP$ property just at the level of filtrations as defined in (5.18). We suppose that $\mathcal{G}$ is the filtration under consideration and $\mathcal{Z}$ is a filtration containing $\mathcal{G}$ and self sufficient in $\mathcal{F}$.

Proposition 5 *$\mathcal{G}$ has the $ANKP$ property with respect to D if and only if every $(\mathcal{Z}, P)$ martingale whose terminal value is $\mathcal{G}_T$ measurable is orthogonal to the martingale $E^P[D_T|\mathcal{Z}_t]$.*

Proof. Let $M(t)$ be a $(\mathcal{Z}, P)$ martingale with terminal value G_T, which is $\mathcal{G}_T$ measurable. By the $ANKP$ property we may write

$$E^Q\left[G_T|\mathcal{Z}_s\right] = E^P\left[G_T|\mathcal{Z}_s\right].$$

The left hand side is given by

$$E^Q\left[G_T|\mathcal{Z}_s\right] = \frac{E^P\left[D_T G_T|\mathcal{Z}_s\right]}{E^P\left[D_T|\mathcal{Z}_s\right]}$$

Substituting above we get that

$$E^P\left[D_T G_T|\mathcal{Z}_s\right] = E^P\left[D_T|\mathcal{Z}_s\right] E^P\left[G_T|\mathcal{Z}_s\right]$$

and the result holds. ■

Proposition 5 describes the precise link between orthogonality and the $ANKP$ property. For example, it is not sufficient to just test the orthogonality of X and D in the case $\mathcal{G}$ is generated by the process X. One needs to consider the martingales in $(\mathcal{Z}, P)$ with $\mathcal{G}_T$ measurable terminal values and ensure that these are all orthogonal to $E^P[D_T|\mathcal{Z}_t]$. Typically the filtration $\mathcal{Z}$ would contain variables necessary for predicting outcomes in $\mathcal{G}$ and the law of these variables may be altered by the change of probability and hence indirectly the law of $\mathcal{G}$ measurable variables. Hence orthogonality tests need to reach beyond just X and D to confirm the $ANKP$ result. However, in many cases it is sufficient to simply check that $E^P[D_T|\mathcal{Z}_t]$ is trivial. This happens because the collection of $(\mathcal{Z}, P)$ martingales finishing in $\mathcal{G}_T$ often generates all the $(\mathcal{Z}, P)$ martingales in the sense of Kunita-Watanabe.

In the special case of $\mathcal{G}$ itself being self sufficient in $\mathcal{F}$ we choose systematically that $\mathcal{Z} = \mathcal{G}$ and we need only consider all the $\mathcal{G}$ martingales in this case. In addition when $\mathcal{G}$ has the martingale representation property with respect to a set of martingales $(\gamma_j, j \in J)$ then one may restrict the orthogonality tests to just these martingales. It follows that if $\mathcal{G}$ has the martingale representation property with respect to X , which is both a $\mathcal{G}$ and $\mathcal{F}$ martingale, then $\mathcal{G}$ is self sufficient in $\mathcal{F}$ and a mere test of the orthogonality of X and D suffices.

It is now helpful to construct a typical situation resulting in the $ANKP$ property for some risk. For this we wish to take for $\mathcal{G}$ the filtration of a risk not self sufficient in $\mathcal{F}$ under P. This means that $\mathcal{F}_s$ contains information about risks in $\mathcal{G}_T$ that $\mathcal{G}_s$ does not have. One of the simplest examples for such a relation between filtrations is that of Brownian bridges whereby we take for $\mathcal{G}$ the filtration of a Brownian motion, $(B_u, u > 0)$, with $\mathcal{G}_t = \sigma(B_u, u \leq t)$ and $\mathcal{G} = \{\mathcal{G}_t, t \leq S\}$. For $T > S$ and for another independent Brownian motion $(W_u, u > 0)$ we let $\mathcal{F}_t = \{\sigma(B(T)) \vee \sigma(B_u, u \leq t) \vee \sigma(W_u, u \leq t)\}$, with $\mathcal{F} = \{\mathcal{F}_t, t \leq S\}$. For the filtration $\mathcal{Z} \supset \mathcal{G}$ self sufficient in $\mathcal{F}$ we take $\mathcal{Z}_t = \{\sigma(B(T) \vee \sigma(B_u, u \leq t)\}$ with $\mathcal{Z} = \{\mathcal{Z}_t, t \leq S\}$.

We now choose the density D to be $\mathcal{F}_S$ measurable with a projection on $\mathcal{Z}$ that is orthogonal to all the $(\mathcal{Z}, P)$ martingales finishing in $\mathcal{G}_S$. For this it is important to know what these martingales are and what are the $(\mathcal{Z}, P)$ martingales orthogonal to all the $(\mathcal{Z}, P)$ martingales finishing in $\mathcal{G}_S$, if any. We now recognize that in this case $\mathcal{Z}_S = \mathcal{G}_S \vee \sigma(B(T) - B(S))$ and $B(T) - B(S)$ is independent of $\mathcal{G}_S$. Hence the projection of D onto $\mathcal{Z}_S$ being $B(T) - B(S)$ measurable will give the required orthogonality of the projection to all the $(\mathcal{Z}, P)$ martingales finishing in $\mathcal{G}_S$. This will deliver the $ANKP$ property of $\mathcal{G}_S$ with respect to D. In particular we may take

$$D = \exp\left(B(T) - B(S) - \frac{T-S}{2}\right) \times \mathcal{E}\left(\int_0^S v(s)dB(s) + w(s)dW(s)\right) \tag{5.27}$$

for any processes $v(s), w(s)$ that are adapted to the $\mathcal{F}$ filtration. The projection of D on $\mathcal{Z}_S$ is then the $\mathcal{Z}_S$ measurable random variable

$$\Delta = \exp\left(B(T) - B(S) - \frac{T-S}{2}\right)$$

whose associated $(\mathcal{Z}, P)$ martingale is orthogonal to all $(\mathcal{Z}, P)$ martingales finishing in $\mathcal{G}_S$ and we therefore have $ANKP$. For the NKP property we take up this example again after we have established the conditions for NKP.

5.4.3 ANKP for filtrations and the examples

We now reexamine the five examples in the light of Proposition 5. For the first example we note that $\mathcal{G}$ is the filtration generated by the Brownian motion X. $\mathcal{G}$ has the martingale representation property with respect to X hence is self sufficient in $\mathcal{F}$ and one only needs to verify the orthogonality of D with respect to X. This example is in line with our final comment in section 4.2.

In the statement of example 1 we can relax the hypothesis that X is a one dimensional Brownian motion and take for X any extremal martingale, which is a martingale for which there is only one equivalent martingale measure with respect to its own filtration $\mathcal{X}$.

In example 2 $\mathcal{G}$ is self sufficient and hence $\mathcal{Z}$ is chosen equal to $\mathcal{G}$. Furthermore $E^P[D_T|\mathcal{G}_t]$ is trivially orthogonal to all the $\mathcal{G}$ martingales and hence we have the $ANKP$ property. We comment for example 3, that the NKP property does not hold and the filtration of X is not self sufficient in $\mathcal{F}$. Indeed there are discontinuous martingales in the $\mathcal{X}$ filtration, Lane (1978) and Knight (1987). In contrast, example 4 is comparable to 3 but the filtration of X is that of $\mathcal{F}$ when the two Brownian motions start away from zero. Example 5 does not quite fit the structure of Proposition 5 as the filtration of X is not self sufficient in $\mathcal{F}$.

5.4.4 A decomposition of risks

In the context of self sufficient filtrations we ask if it is possible to decompose all risks viewed as $\mathcal{F}$ measurable martingales as the sum of two risk classes, those that are not kernel priced with a restricted filtration $\mathcal{G}$ and have the $ANKP$ property, and those that are kernel priced with the restricted filtration $\mathcal{H}$. A more precise statement follows below.

We say that two filtrations $\mathcal{G}$ and $\mathcal{H}$ *supplement* each other in $\mathcal{F}$ if both are self sufficient, every $\mathcal{G}$ martingale is orthogonal to every $\mathcal{H}$ martingale, and every square integrable $\mathcal{F}$ martingale is generated in the sense of Kunita and Watanabe by the square integrable martingales with respect to either $\mathcal{G}$ or $\mathcal{H}$.

Furthermore we say that a *pricing decomposition* of $\mathcal{F}$ risks with respect to D exists if there exist two filtrations $\mathcal{G}$ and $\mathcal{H}$ supplementing each other in $\mathcal{F}$ with the further requirement that every $\mathcal{G}$ martingale has the $ANKP$ property and no nontrivial $\mathcal{H}$ martingale has the $ANKP$ property.

We present some partial results in this direction at the level of square integrable martingales. We begin by selecting $\mathcal{G}^*$ a maximal self sufficient filtration with the $ANKP$ property. We then define by $\mathcal{M}_{\mathcal{G}^*}$ the collection of all square integrable $\mathcal{G}^*$ martingales. We construct from $\mathcal{M}_{\mathcal{G}^*}$ the $\mathcal{F}$ stable subspace $\mathcal{S}_2(\mathcal{M}_{\mathcal{G}^*})$ generated by these martingales in $\mathcal{M}_{\mathcal{F}}$. We further observe that $\mathcal{S}_2(\mathcal{M}_{\mathcal{G}^*})$ is orthogonal to the change of measure D. Hence the orthogonal complement of $\mathcal{S}_2(\mathcal{M}_{\mathcal{G}^*})$ in the Hilbert space $\mathcal{M}_{\mathcal{F}}$ is nonempty. We may therefore decompose the Hilbert space $\mathcal{M}_{\mathcal{F}}$ as

$$\mathcal{M}_{\mathcal{F}} = \mathcal{S}_2(\mathcal{M}_{\mathcal{G}^*}) \oplus \mathcal{K}.$$

It remains to relate $\mathcal{K}$ to the stable space generated by a set of priced risks. A more constructive characterization of the subspace of priced risks is left for future research at this stage.

5.5 Self Sufficiency and Measure Changes

In this section we study conditions on the change of measure density with the property that self sufficient filtrations under P remain self sufficient under Q. As we have noted in Proposition 2, this property is essential to deliver the NKP property from the $ANKP$ property that we have studied in detail. For simplicity we assume throughout that all $(\mathcal{F}\ , P)$ martingales are continuous. We introduce a collection of processes associated with the change of measure density D and the filtration $\mathcal{J}$ which we suppose is self sufficient under P in $\mathcal{F}$.

The first process we identify is the projection of D onto the filtration $\mathcal{J}$ that we define by Δ,

$$\Delta(t) = E^P \left[D(t) | \mathcal{J}_t \right].$$

We next define the stochastic logarithms of D, Δ by L, Λ as

$$\begin{aligned} L(t) &= \int_0^t \frac{dD(s)}{D(s)} \\ \Lambda(t) &= \int_0^t \frac{d\Delta(s)}{\Delta(s)} \end{aligned}$$

so that $D(t) = \mathcal{E}(L)_t$, $\Delta(t) = \mathcal{E}(\Lambda)_t$.

Proposition 6 *Assume that $\mathcal{J}$ is self sufficient in $\mathcal{F}$ under P, then $\mathcal{J}$ is self sufficient in $\mathcal{F}$ under Q if and only if the $(\mathcal{F}\ , P)$ local martingale*

$$L - \Lambda$$

is orthogonal to the set of all $(\mathcal{J}\ , P)$ local martingales. Equivalently the Kunita-Watanabe projection of L on $(\mathcal{J}\ , P)$ is Λ.

Proof. We simultaneously establish both directions. A classical reinforcement of Girsanov's theorem states that every $(\mathcal{J}\ , Q)$ local martingale $\widetilde{I}^Q$may be obtained as

$$\widetilde{I}^Q(t) = I^P(t) - \int_0^t \frac{d\langle I^P, \Delta\rangle_u}{\Delta(u)} \tag{5.28}$$

for I^P a generic $(\mathcal{J}\ , P)$ local martingale.

Since under self sufficiency of $\mathcal{J}$ in $\mathcal{F}$ under P, I^P is also a $(\mathcal{F}, P)$ local martingale we may write, by using Girsanov's theorem with respect to the filtration $\mathcal{F}$ that

$$I^P(t) = J^Q(t) + \int_0^t \frac{d\langle I^P, D\rangle_u}{D(u)} \tag{5.29}$$

where $(J^Q(t), t \geq 0)$ is a $(\mathcal{F}, Q)$ local martingale.

Combining (5.28) and (5.29) we obtain

$$\begin{aligned} \widetilde{I}^Q(t) &= J^Q(t) + \int_0^t \frac{d\langle I^P, D\rangle_u}{D(u)} - \int_0^t \frac{d\langle I^P, \Delta\rangle_u}{\Delta(u)} \\ &= J^Q(t) + \int_0^t d\langle I^P, L\rangle_u - \int_0^t d\langle I^P, \Lambda\rangle_u \end{aligned}$$

Thus $\widetilde{I}^Q(t)$ is a $(\mathcal{F}, Q)$ local martingale if and only if

$$\int_0^t d\langle I^P, L\rangle_u - \int_0^t d\langle I^P, \Lambda\rangle_u = 0$$

which proves the result. ■

Two applications of Proposition 5, in the one and two dimensional cases illustrate the type of measure changes preserving self sufficiency of a filtration.

Example 7 *Assume $\mathcal{J}$ is self sufficient in $\mathcal{F}$ under P and that:*

· every $(\mathcal{F}, P)$ martingale is a stochastic integral with respect to $(B(t), t \geq 0)$ a one dimensional Brownian motion

· every $(\mathcal{J}, P)$ martingale is a stochastic integral with respect to $(\beta(t), t \geq 0)$ a one dimensional $(\mathcal{J}, P)$ Brownian motion

Then, $\mathcal{J}$ is self sufficient in $\mathcal{F}$ under Q if and only if

$$D = \Delta \text{ or equivalently } L = \Lambda.$$

Proof. Since $L - \Lambda$ is orthogonal to all the $(\mathcal{J}, P)$ martingales and since under the joint three hypotheses these martingales generate in the sense of Kunita, all the $(\mathcal{F}, P)$ martingales it must be that $L - \Lambda = 0$. ■

A concrete example is given by $\mathcal{F}$ the natural filtration of a one dimensional Brownian motion $(B(t), t \geq 0)$, and $\mathcal{J}$ the filtration of the absolute value of $(|B(t)|, t \geq 0)$. It is well known that $\mathcal{J}$ is self sufficient in $\mathcal{F}$ under P and that the $(\mathcal{J}, P)$ Brownian motion may be chosen such as

$$\beta(t) = \int_0^t sign(B(s))dB(s).$$

If we take for D

$$D(t) = \exp\left(\mu B(t) - \frac{\mu^2 t}{2}\right)$$

then $\mathcal{J}$ is not self sufficient in $\mathcal{F}$ under Q. Here we must have the density adapted to $\mathcal{J}$ for stability of self sufficiency under a change of measure. It is not so adapted. More precisely we may evaluate

L and Λ as follows. First we obtain that

$$\begin{aligned}\Delta(t) &= \cosh(\mu B(t))e^{-\frac{\mu^2 t}{2}} \\ &= \exp\left(\log\left(\cosh\left(\mu B(t)\right)\right) - \frac{\mu^2 t}{2}\right)\end{aligned}$$

Thus

$$\begin{aligned}L(t) &= \mu B(t) \\ \Lambda(t) &= \mu\int_0^t \tanh\left(\mu B(s)\right) dB(s) \\ &= \mu\int_0^t \tanh(\mu|B(s)|)d\beta(s)\end{aligned}$$

and we explicitly see that $L \neq \Lambda$.

Example 8 *Assume $\mathcal{J}$ is self sufficient in $\mathcal{F}$ under P and that*

· every $(\mathcal{J}, P)$ martingale is a stochastic integral with respect to $(\beta(t), t \geq 0)$ a one dimensional $(\mathcal{J}, P)$ Brownian motion

· every $(\mathcal{F}, P)$ martingale is a stochastic integral with respect $(B(t), t \geq 0)$ a two dimensional $(\mathcal{F}, P)$ Brownian motion $((\beta(t), \gamma(t)); t \geq 0,$

Then, $\mathcal{J}$ is self sufficient in $\mathcal{F}$ under Q if and only if for some $\mathcal{F}$ predictable process $l_2(s)$

$$L(t) - \Lambda(t) = \int_0^t l_2(s)d\gamma(s)$$

for some $(l_2(s), s \geq 0)$ a $\mathcal{F}$ predictable process such that

$$\int_0^t (l_2(s))^2\, ds < \infty$$

and in this case we have

$$\begin{aligned}\Lambda(t) &= \int_0^t \lambda(s)d\beta(s) \text{ and} \\ L(t) &= \int_0^t \lambda(s)d\beta(s) + l_2(s)d\gamma(s)\end{aligned} \tag{5.30}$$

Proof. The representation for $L(t)$ in the form

$$L(t) = \int_0^t l_1(s)d\beta(s) + \int_0^t l_2(s)d\gamma(s)$$

is immediate. In order that $L - \Lambda$ is orthogonal to the set of all $(\mathcal{J}, P)$ martingales it is necessary and sufficient that

$$\langle L - \Lambda, \int_0^t \left(l_1(s) - \lambda(s)\right) d\beta(s)\rangle = 0$$

or that $l_1(s) = \lambda(s)$ $dsdP$ almost surely. ■

For a concrete example we take for $\mathcal{F}$ the natural filtration of $(U(t), V(t); t \geq 0)$ a two dimensional Brownian motion and $\mathcal{J} = \{\mathcal{J}_t : t \geq 0\}$ the filtration of

$$\left(R^2(t) = U^2(t) + V^2(t), t \geq 0\right)$$

Again it is well known that $\mathcal{J}$ is self sufficient in $\mathcal{F}$ under P and that the Brownian motion

$$\beta(t) = \int_0^t \frac{U(s)dU(s) + V(s)dV(s)}{R(s)}, t \geq 0$$

generates all the $(\mathcal{J}\ , P)$ martingales. We may also take the second Brownian motion

$$\gamma(t) = \int_0^t \frac{V(s)dU(s) - U(s)dV(s)}{R(s)}$$

and it is well known that β and γ, which is independent of β, generate all the $(\mathcal{F}\ , P)$ martingales.

Consider for D

$$D(t) = \exp\left(a\int_0^t R(s)d\gamma(s) - \frac{a^2}{2}\int_0^t R^2(s)ds\right)$$

so that now we have

$$\begin{aligned} L(t) &= a\int_0^t R(s)d\gamma(s) \\ \Delta(t) &= E^P[D(t)|\mathcal{J}_t] \equiv 1 \end{aligned}$$

or that $\Lambda(t) \equiv 0$, and $L(t) = L(t) - \Lambda(t)$ is orthogonal to the $(\mathcal{J}\ , P)$ martingales. Hence we still have $\mathcal{J}$ self sufficient in $\mathcal{F}$ under Q.

Combining the lessons of examples 7 and 8 we learn that to preserve self sufficiency, the change of measure density factors as

$$D(t) = \Delta(t)\mathcal{E}(L - \Lambda)_t.$$

where $L - \Lambda$ is orthogonal to all the $(\mathcal{J}\ , P)$ martingales and in particular to Λ.

We consider now the density (5.27) in our example for $ANKP$ using the Brownian bridge. We see that for the orthogonality of $L - \Lambda$ to all the $(\mathcal{Z}, P)$ martingales we cannot have the first integral with respect to the Brownian motion $B(s)$ and to preserve the self sufficiency of $\mathcal{Z}$ in $\mathcal{F}$ under Q we must take the density in the form

$$D = \exp\left(B(T) - B(S) - \frac{T - S}{2}\right) \times \mathcal{E}\left(\int_0^S w(s)dW(s)\right)$$

with $w(s)$ adapted to the $\mathcal{F}$ filtration. Financially we have that the risk of the Brownian motion $B(u), u \leq S$ is not priced. An independent component in $\mathcal{Z}$ is priced as is the Brownian motion $W(u), u \leq S$ with the prices being $\mathcal{F}$ adapted. In this case we also have NKP.

To illustrate what is necessary for the failure of the self sufficiency of $\mathcal{J}$ in $\mathcal{F}$ under Q we observe that, in the context of example 8, using Proposition 6,

$$L(t) - \Lambda(t) = \int_0^t l_1(s)d\beta(s) + \int_0^t l_2(s)d\gamma(s)$$

for some particular integrand $l_1(s)$. Indeed, since the projection onto the filtration generated by β of the integral of l_1 with respect to β is zero it must be the case that the integrand is special and in fact

$$E\left[l_1(s)|\mathcal{B}_\infty\right] = 0$$

To establish this result let

$$M_t^u = \int_0^t u_s d\beta_s$$

for any u_s predictable with respect to $\mathcal{F}$ and $E\left[\int_0^t u_s^2 ds < \infty\right]$. It is a simple consequence of the martingale representation property for $(\mathcal{J}, P)$ that

$$E\left[M_t^u|\mathcal{J}_t\right] = \int_0^t E\left[u_s|\mathcal{J}_s\right] d\beta_s$$

consequently $E\left[M_t^u|\mathcal{J}_t\right] = 0$ if and only if $E\left[u_s|\mathcal{J}_\infty\right] = 0$.

The density D has in this case a sort of hidden exposure to the risks of filtration $\mathcal{J}$ in that conditional on all of $\mathcal{J}_\infty$ these exposures cannot be observed by evaluating conditional expectations of the integrand of exposure. The density would have to be specially constructed with respect to the particular risk filtration to mask the conditional expectation of the exposure. Barring the presence of such hidden exposures we expect to maintain self sufficiency of $\mathcal{J}$ in $\mathcal{F}$ under Q when we have it under P.

5.6 Event Risk and No Kernel Price

One is often interested in the pricing of events like default or first passage to various boundaries. In fact we now have specific event risks with a growing market like the credit default swap contracts. In this section we inquire about the no price property for events and the martingales associated with them via conditional expectations. Consider a set $A \in \mathcal{F}_T$ and the associated $(\mathcal{F}, P)$ martingale

$$X^A(t) = E^P\left[\mathbf{1}_A|\mathcal{F}_t\right].$$

Unlike the martingales considered earlier, orthogonality of X^A and D ensures that there is no change of probability for the event risk or that

$$Q(A) = P(A).$$

Orthogonality does ensure no risk premium on the event risk security itself. This is a consequence of the following equalities.

$$\begin{aligned}
E^Q\left[\mathbf{1}_A\right] &= E^P\left[D(T)\mathbf{1}_A\right] \\
&= E^P\left[X^A(T)D(T)\right] \\
&= E^P[X^A(0)D(0) + \langle X^A, D\rangle_T] \\
&= E^P[X^A(0)] + E^P\left[\langle X^A, D\rangle_T\right] \\
&= E^P[X^A(T)] + E^P\left[\langle X^A, D\rangle_T\right] \\
&= E^P\left[\mathbf{1}_A\right] + E^P\left[\langle X^A, D\rangle_T\right].
\end{aligned}$$

However even in the presence of zero correlation continuously in that $\langle X^A, D\rangle_t \equiv 0$ we may have the absence of the NKP property for the martingale X^A. Financially we may be interested in the structure of excess returns on derivatives written on the price process

$$J^A(t) = E^Q\left[\mathbf{1}_A|\mathcal{F}_t\right]$$

We show here that such derivatives may experience a non zero excess return even under orthogonality and even when $J^A(t) = X^A(t)$ and there is no risk premium on the base security.

We give in this section two examples of this situation. The first is in a two dimensional Brownian setting while the second considers the one dimensional case. The design of these examples is comparable and we present their structure together. Table 1 below presents the choice of D and the set A for both contexts.

TABLE 1
Structure of Event Risk Examples

	2 Dimensions	1 Dimension
$D(t) = \mathcal{E}(L)_t$	$L(t) =$	$L(t) =$
	$\int_0^t U(s)dU(s) + V(s)dV(s)$	$\lambda\int_0^t \mathbf{1}_{B(s)>0}dB(s)$
	$= \frac{1}{2}\left(R^2(t) - 2t\right)$	$= \lambda\left(B(t)^+ - \frac{1}{2}\Sigma(t)\right)$
$A = (\widehat{\gamma}_T > 0)$	$\Theta(t) \underset{Def}{=}$	$\Theta(t) \underset{Def}{=}$
with $\widehat{\gamma}$ a Brownian	$\int_0^t -U(s)dV(s) + V(s)dU(s)$	$B(t)^- - \frac{1}{2}\Sigma(t)$
motion defined via:	$= \widehat{\gamma}\left(\int_0^t R(s)^2 ds\right)$	$= \widehat{\gamma}\left(\int_0^t \mathbf{1}_{B(s)<0}ds\right)$

In both cases, we first show that $X^A(t)$ may be represented as a stochastic integral with respect to $d\Theta$ which is orthogonal to L. For this purpose we begin to express $\mathbf{1}_{\widehat{\gamma}(T)>0}$ as a stochastic integral with respect to $d\widehat{\gamma}$. Specifically we write

$$\mathbf{1}_{\widehat{\gamma}(T)>0} = \frac{1}{2} + \int_0^T \phi(s, \widehat{\gamma}(s))d\widehat{\gamma}(s)$$

where

$$\phi(s,x) = \frac{1}{\sqrt{T-s}} n\left(\frac{x}{\sqrt{T-s}}\right)\mathbf{1}_{s<T}$$

The computation of ϕ is comparable to the one of $q(s,x)$ in Example 2. Making the change of variable $s = H(u)$ where in the 2 dimensional case

$$H(u) = \int_0^u R^2(v)dv$$

and in the one dimensional case

$$H(u) = \int_0^u \mathbf{1}_{B(v)<0}dv$$

we obtain

$$\mathbf{1}_{\widehat{\gamma}(T)>0} = \frac{1}{2} + \int_0^\infty \phi(H(u), \widehat{\gamma}(H(u)))\mathbf{1}_{H(u)<T}d\widehat{\gamma}(H(u))$$

which provides us with a formula for $(X^A(t), t \geq 0)$ in both cases:

$$X^A(t) = \frac{1}{2} + \int_0^t \phi(H(u), \Theta(u))\mathbf{1}_{H(u)<T} d\Theta(u)$$

In order to show that the martingale X^A fails the NKP property it remains to show that the laws of $\left(\phi^2(H(u), \Theta(u))\mathbf{1}_{H(u)<T} \frac{d}{du} (\langle\Theta\rangle(u)), u \geq 0\right)$ under Q and under P differ. Some elementary manipulations reduce the problem to showing that the laws of $(H(u), u \geq 0)$ differ under P and Q and this is easily verified and was referred to in previous examples.

5.7 Conclusion

This paper documents the problems inherent in viewing risk pricing in purely covariation terms. We provide examples of processes correlated with the change of measure density for which there is no change of probability on the reduced self sufficient filtration containing the risk being considered. Hence there is no excess return on derivative claims written on these processes provided the self sufficient filtration under P remains self sufficient under Q. This property is called almost no kernel pricing, $ANKP$ as it falls just short of NKP which also requires the invariance of self sufficiency to the change of measure. However, we show that the loss of self sufficiency under Q when we have it under P requires a very specially designed kernel with exposure to the martingales of the self sufficient filtration that vanish on projection onto this filtration and are therefore a form of hidden exposures. We also provide examples with zero correlation where the risks involved are priced and we have excess returns associated with the investments involved.

These examples lead us to investigate the precise form of no kernel pricing in terms of correlation considerations. There are two sets of correlation conditions involved. The first delivers $ANKP$ and introduces the concept of the self sufficient filtration containing a particular risk filtration. The self sufficient filtration heuristically includes all variables useful in predicting the future path of the risk in question. Hence we have the smaller filtration which we call the risk filtration that is contained in the larger self sufficient filtration containing the risk filtration that plays a critical role in understanding the pricing of risks. For $ANKP$ to hold, the projection of the pricing kernel onto the self sufficient filtration must be orthogonal to all the martingales in the self sufficient filtration that end in a random variable measurable with respect to the smaller risk filtration.

The second set of correlation conditions are designed to ensure that self sufficiency is maintained under the measure change, a property critical for $ANKP$ to actually yield NKP. For NKP it is required that the stochastic logarithm of the density minus the stochastic logarithm of the projection be orthogonal to all the martingales in the self sufficient filtration. Equivalently, the projection of the difference must vanish and hence we call these exposures hidden exposures.

Additionally we show that the density is strongly negatively priced in that, for example, all monotonic increasing functions of the density receive a negative risk premium. Furthermore, the equivalence of the density under increasing monotonic transformations is the unique random variable with this property. It is therefore the classic insurance asset.

A number of interesting questions remain. We would like to know if there exists a change of measure in a continuous time setting with the property that all nontrivial martingales on the filtration of the economy experience a change of probability or that all risks are priced. For the case of a one factor model with a constant price for the Brownian motion risk (Example 5) we showed that there exists a semimartingale that undergoes no change of probability. It remains to find a martingale if any, with this property.

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